DOCTORADO EN
MATEMÁTICA APLICADA

Facultad de Ciencias
Universidad del Bío-Bío

Programa de Doctorado en Matemática Aplicada

UNIVERSIDAD DEL BÍO-BÍO

Conforming and non-conforming discretizations for desalination process models

Thesis submitted to Universidad del Bío-Bío in fulfillment of the requirements for the degree of Doctor en Matemática Aplicada

by Miguel Serón Almonacid

Director: Ricardo Oyarzúa Vargas

Co-director: Jessika Camaño Valenzuela,
Manuel Solano Palma.

Agradecimientos

A mi pareja, Marcela, por su amor, comprensión y compañía durante todo este proceso.

A mis abuelos que ya partieron, a mis padres, tíos, familia y amigos en general, que cada uno de ellos ha sido un apoyo vital e importante de este gran proceso.

Agradecer a mis directores de tesis Dr. Ricardo Oyarzúa, Dra. Jessika Camaño y Dr. Manuel Solano, por aceptar guiarme en el camino para desarrollar esta tesis, por la paciencia y disposición mostrada en cada ocasión y, por todo el conocimiento entregado en mi formación.

Al Dr. Sander Rhebergen, por su recepción y apoyo durante la pasantía desarrollada en la University of Waterloo, Canadá.

Quisiera agradecer también a Dr. David Mora y Dr. Pablo Venegas, por lo enseñado y aprendido en las distintas instancias que participaron de mi formación y por su buena disposición para ser evaluadores en mi Examen de Calificación y defensa de Proyecto de Tesis.

A Nicolás Carro y Fernando Gajardo, por el apoyo técnico brindado y su contribución en ayudarme a comprender mejor los procesos y fenómenos relacionados con la desalinización.

A Ginette Rivas, Secretaria del Doctorado en Matemática Aplicada, agradecer su disponibilidad ante cualquier duda o requerimiento que pudo haberse presentado.

Finalmente, agrader a las instituciones y programas que han apoyado en mi formación: la Agencia Nacional de Inverstigación y Desarrollo (ANID-Chile) a través de Becas/Doctorado nacional 21190766, Anillo of Computational Mathematics for Desalination Processes (ACT210087).

Miguel Serón A.

Resumen

El objetivo de esta tesis es proponer y analizar varios esquemas de elementos finitos para un problema de interacción fluido-membrana no isotérmico que surge en los procesos de destilación por membrana. El problema consiste en un sistema de Navier–Stokes/calor, comúnmente conocido como el sistema de Boussinesq, en la región de fluido libre, y un sistema acoplado Darcy-calor en la membrana. Estos sistemas están acoplados a través de términos de flotabilidad y un conjunto de condiciones de transmisión en la interfaz fluido-membrana, incluyendo la conservación de masa, balance de fuerzas normales, la ley de Beavers–Joseph–Saffman y la continuidad del flujo de calor y la temperatura del fluido.

En primer lugar, proponemos y analizamos un método de elementos finitos conforme para el problema de interacción fluido-membrana no isotérmico. Consideramos la formulación variacional estándar de velocidad-presión-temperatura para el sistema de Boussinesq, junto con un esquema dual-mixto acoplado con una formulación primal para las ecuaciones de Darcy y calor en la región de la membrana. Este enfoque introduce la traza de la presión del medio poroso como un multiplicador de Lagrange adecuado. Probamos la existencia de una solución continua mediante una estrategia de punto fijo y bajo una suposición de datos suficientemente pequeños, mientras que para la prueba de unicidad se requiere una suposición restrictiva adicional sobre la solución de la temperatura en la membrana. Para el esquema de Galerkin asociado, empleamos elementos de Bernardi–Raugel y Raviart–Thomas para las velocidades, elementos constantes a trozos para las presiones y funciones lineales continuas a trozos para las temperaturas y el multiplicador de Lagrange en una partición de la interfaz. Probamos que el problema esta bien puesto tanto para los esquemas continuo como discreto y establecemos las correspondiente estimación de error. Se presentan ejemplos numéricos para confirmar las tasas de convergencia predichas y demostrar el rendimiento del método.

En segundo lugar, desarrollamos un análisis de error a posteriori para el problema de interacción fluido-membrana no isotérmico conforme. Para derivar el estimador de error a posteriori, utilizamos argumentos estándar basados en técnicas de dualidad, descomposiciones de Helmholtz y propiedades de aproximación local de los operadores de interpolación de Raviart–Thomas y Clément. Además,

se emplean desigualdades inversas, técnicas de localización usuales de funciones burbuja y resultados conocidos de trabajos previos para probar la eficiencia local del estimador de error a posteriori propuesto. Proponemos un algoritmo adaptativo basado en un estimador de error a posteriori confiable y eficiente.

Por último proponemos y analizamos un método de elementos finitos que conserva masa para el mismo problema de interacción fluido-membrana no isotérmico. Consideramos un esquema variacional de velocidad-presión-temperatura para el sistema de Boussinesq en la región de fluido libre, mientras que en la región de la membrana utilizamos una formulación dual-mixta para el sistema de Darcy acoplado con una ecuación primal para el modelo de temperatura. Esta formulación resulta en incógnitas dadas por la velocidad, la presión y la temperatura en ambos dominios. Para el esquema de Galerkin asociado, combinamos un esquema $\mathbf{H}(\mathrm{div})-$conforme para las variables del fluido y una discretización conforme de Galerkin para la ecuación de calor. El esquema numérico resultante produce velocidades de divergencia libre y preserva la ley de conservación de la masa a nivel discreto. El análisis de los problemas continuos y discretos se lleva a cabo utilizando una estrategia de punto fijo bajo una suposición de datos suficientemente pequeños. Obtenemos estimaciones de error óptimos bajo una suposición adicional sobre los datos y presentamos resultados numéricos que ilustran el rendimiento del método.

Palabras Claves: Fluido-membrana no isotérmico, sistema acoplado Navier–Stokes/Darcy/calor, método de elementos finitos mixto, conservador de masa, métodos de Galerkin discontinuo, análisis de error a priori, estimador de error a posteriori.

Abstract

The aim of this thesis is to propose and analyze several finite element schemes for a nonisothermal fluid-membrane interaction problem that arises in membrane distillation processes. The problem consists of a Navier–Stokes/heat system, commonly known as the Boussinesq system, in the free-fluid region, and a Darcy-heat coupled system in the membrane. These systems are coupled through buoyancy terms and a set of transmission conditions on the fluid-membrane interface, including mass conservation, balance of normal forces, the Beavers–Joseph–Saffman law, and continuity of heat flux and fluid temperature.

Firstly, we propose and analyze a conforming finite element method for the nonisothermal fluid-membrane interaction problem. We consider the standard velocity-pressure-temperature variational formulation for the Boussinesq system, along with a dual-mixed scheme coupled with a primal formulation for the Darcy and heat equations in the membrane region. This approach introduces the trace of the porous medium pressure as a suitable Lagrange multiplier. We prove the existence of a continuous solution by means of a fixed-point strategy and under a smallness assumption on data, while for the uniqueness proof, an additional restrictive assumption on the membrane temperature solution is required. For the associated Galerkin scheme, we employ Bernardi–Raugel and Raviart–Thomas elements for velocities, piecewise constant elements for pressures, and continuous piecewise linear functions for the temperatures and the Lagrange multiplier on a partition of the interface. We prove well-posedness for both the continuous and discrete schemes and derive corresponding error estimates. Numerical examples are presented to confirm the predicted convergence rates and demonstrate the performance of the method.

Secondly, we develop an a posteriori error analysis for the conforming nonisothermal fluid-membrane interaction problem. To derive the a posteriori error estimator, we utilize standard arguments relying on duality techniques, Helmholtz's decompositions, and local approximation properties of the Raviart–Thomas and Clément interpolation operators. Additionally, inverse inequalities, localization techniques of bubble functions, and known results from previous works are employed to prove the local efficiency of the proposed a posteriori error estimator.

We propose an adaptive algorithm based on a reliable and efficient a posteriori error estimator.

Finally, we propose and analyze a mass-conservative finite element method for the same nonisothermal fluid-membrane interaction problem. We consider a velocity-pressure-temperature variational scheme for the Boussinesq system in the free-fluid region, while in the membrane region, we use a dual-mixed formulation for the Darcy system coupled with a primal equation for the temperature model. This formulation results in unknowns given by the velocity, pressure, and temperature in both domains. For the associated Galerkin scheme, we combine an $\mathbf{H}(\mathrm{div})$-conforming scheme for the fluid variables and a conforming Galerkin discretization for the heat equation. The resulting numerical scheme produces exactly divergence-free velocities and preserves the law of conservation of mass at a discrete level. The analysis of the continuous and discrete problems is carried out using a fixed-point strategy under a sufficiently small data assumption. We derive optimal error estimates under an additional assumption over the data and present numerical results illustrating the performance of the method.

Key Words: nonisothermal fluid-membrane, Navier–Stokes/Darcy/heat coupled system, mixed finite element method, mass-conservative, Discontinous Galerkin methods, a priori error analysis, a posteriori error estimator.

Contents

Chapter 1

Introduction

1.1 Motivation

Access to fresh and potable water is a critical global challenge, particularly in arid and water-scarce regions, which is why many countries in the Middle East, Southeast Asia and North Africa are now classified as water-scarce regions (see [78]). However, this problem has spread to different regions where this phenomenon of shortages had not been a big problem until a couple of years ago. Particularly in Chile, the situation is critical since it will be among the 30 most water-stressed countries in 2040 according by the World Resources Institute (see [77]). Consequently, it is imperative to investigate innovative methodologies to ensure water resources for human consumption, agricultural irrigation, mining industry, among others.

Throughout human history, various civilizations have devised numerous methods and developed innovative techniques to harness the natural resources available in their environment to supply the necessary amounts of water for human settlements. Notable examples include rainwater harvesting, artificial aquifer recharge, the construction of aqueducts, reservoirs, and dams, as well as the transfer of water between basins.

However, there are two sustainable and reliable methods that allow us to increase the water supply beyond what is available in the natural hydrological cycle: water reuse and desalination. The first process involves the treatment of wastewater to remove contaminants until it meets the relevant standards and criteria of suitability for human consumption, while the second is a process related to the removal of mineral components from saline water.

In particular, desalination has emerged as a promising solution to address the increasing demand for freshwater resources. This technology can be categorized into two main processes: thermal and membrane processes. Thermal desalination separates salt from water through evaporation and condensation, whereas membrane desalination uses a membrane through which water diffuses while salts are retained on the feed side of the membrane (see [82]).

On a large scale, the initial installation of desalination plants commenced in the Middle East during the 1950's ([66]). Today, according to the International Energy Agency (IEA), approximately 58% of the world's total desalination capacity is located in the combined regions of the Middle East and North Africa (see Figure 1.1), while globally, around 90 million m^3 of water are desalinated every day. Thermal desalination plants dominate the desalination industry in the Middle East and constitute around 70% of the total desalination operations in the Gulf region. However, thermal desalination is more expensive from an energy and plant maintenance point of view compared to membrane processes such as reverse osmosis (RO). Therefore, reverse osmosis desalination installations have been gradually increasing, comprising around 80% of the total current desalination plants worldwide.

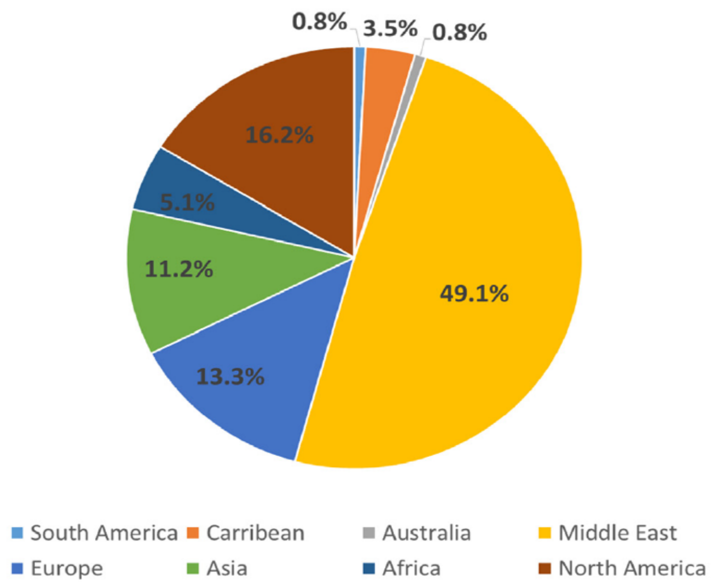

Figure 1.1: Worldwide desalination capacity by region (see [1]).

Although reverse osmosis (RO) is the most widely adopted technique in the industry today, the associated high greenhouse gas emissions, the generation of large quantities of brine containing chemicals, and the significant consumption of electricity used, drives the development of alternative techniques such as: Forward Osmosis (FO), Membrane Distillation (MD), Membrane Capacitive Deionisation

(MCDI). These emerging technologies offer significant advantages over reverse osmosis, such as higher salt rejection content (MCDI, MD), higher water recovery (MD), fewer pretreatment steps (MD, FO), and the ability to use low energy quality (MD, FO). However, standalone technologies still have difficulty competing with RO until specific conditions are met, so the study of these or other technologies is important to improve the efficiency in which water is obtained through desalination processes. Figure 1.2, taken from [98], provides a qualitative comparison of RO, FO, MD, MCDI and their hybrid counter-parts. There, it can be seen that while RO has a low water cost, understood as the produced water is economically viable and cost-competitive with current technologies, it has significant energy requirements and generates substantial brine and greenhouse gas emissions. In contrast, MD reduces energy consumption by facilitating the integration of renewable energies, resulting in a lower environmental impact and reduced waste. Furthermore, by combining both processes (RO + MD) and decreasing both the cost competitiveness of water and the use of renewable energy, higher water recovery can be achieved.

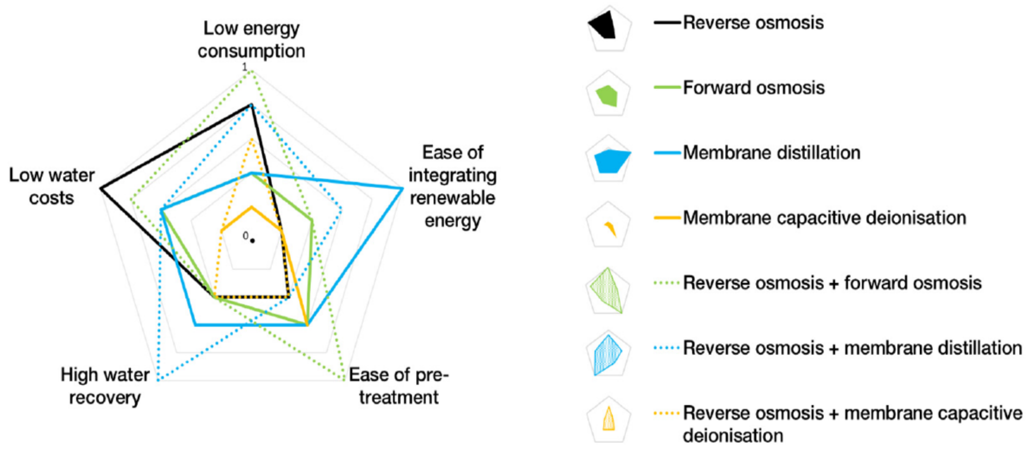

Figure 1.2: Qualitative comparison of RO, FO, MD, MCDI and their hybrid counterparts.

Regarding the physical-chemical processes used in the RO and MD methods, in the first one, the seawater is pressurized and forced through a semi-permeable membrane, allowing water molecules to pass while blocking the passage of dissolved salts and contaminants. The applied pressure exceeds the osmotic pressure, driving the water against its natural flow and resulting in the production of freshwater on one side of the membrane. On the other hand, in MD processes, a saline solution is brought into contact with one side of the membrane, while a low-pressure vapor phase, typically created by applying a temperature gradient, exists on the opposing side. The hydrophobic nature of the membrane allows only water vapor to permeate through, effectively preventing the passage of salts and other impurities. Subsequently, the vapor is condensed on the colder side of the membrane, resulting in the generation of freshwater.

Membrane distillation (MD) encompasses various configurations, each designed to optimize specific desalination goals (see [38]). These configurations are described below, and Figure 1.3 illustrates each respective design:

- Direct contact membrane distillation (DCMD), where a saline solution and a cold condensing fluid come into direct contact with opposing sides of the hydrophobic membrane. The vapor generated on the hot side permeates through the membrane, and upon condensation, fresh water is collected.

- Air gap membrane distillation (AGMD), by providing a stagnant air space in the permeate channel between the membrane and one surface of a condensation plate while the other surface of the condensation plate is to be cooled by cold circulating flow. Volatile molecules evaporate at the feed-membrane interface, transfer through the membrane and the air gap and condense on the cooling plate. The condensate then exits the MD module under the influence of gravity.

- Vacuum membrane distillation (VMD), applies vacuum on the permeate side of the membrane module, created by a vacuum pump. This causes the volatile molecules to evaporate from the feed side of the membrane and transfer through the membrane pores to the permeate channel, where they condense outside the MD module. This configuration enhances separation efficiency and is used in desalination and purification applications.

- Sweeping gas membrane distillation (SGMD), an inert gas (usually air) with a small or negligible concentration of the volatile components is sweeping through the permeate channel of the MD module. Determined by the concentration of volatile molecules, the partial pressure of these components at the permeate-membrane interface is less than the saturation pressure at the feed-membrane interface. Therefore, the volatile molecules evaporate at the feed-membrane interface through the membrane pores, transfer through the membrane and sweep through the permeate condensing outside of the MD module.

In particular, DMCD is the most studied configuration of membrane distillation. Some of the earliest studies dedicated to modeling this configuration were

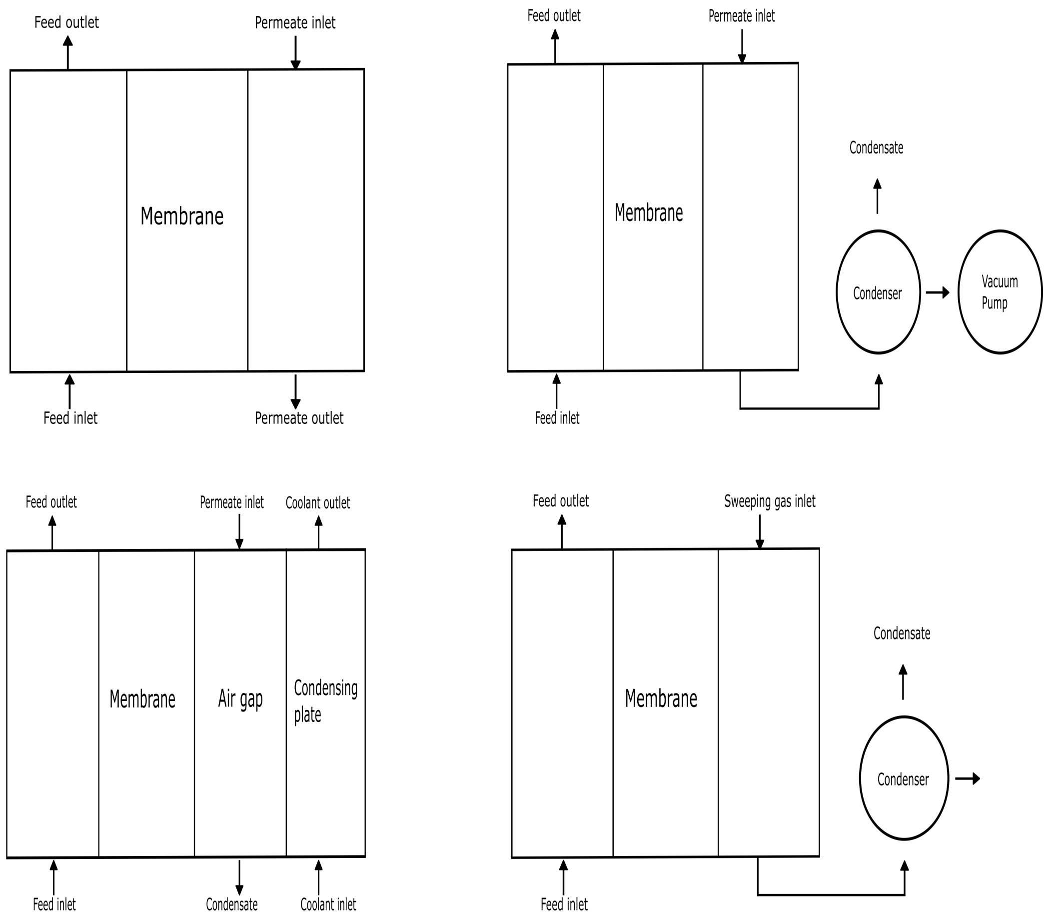

Figure 1.3: Configurations of Membrane Destilation models. In first row DCMD (left) and VMD (right), in second row AGMD (left) and SGMD (right).

the pioneering works of [46, 65, 72, 95], developed in the 1980's. These studies employed integral equations and empirical correlations to describe heat transfer across the membrane, mass balance in the membrane, and the possible types of transmembrane flux (viscous flow, Knudsen flow, and molecular diffusion). However, these studies did not describe the mass fluxes of the feed and permeate, and the feed and permeate domains were only characterized by heat fluxes and the temperatures of the bulk phase and boundary layer. Subsequently, in [74], the authors complemented the existing models and improved the accuracy of the calculations by proposing a model that focused on the feed and permeate domains, providing heat balance, mass balance, and fluid dynamics for these domains. In the last decade, due to rapid advancements in technology and significant scientific attention on simulation software, new models using computational fluid dynamics (CFD) analysis have been presented [67, 97, 102, 103], employing partial differential equations (PDE) to describe the mass, heat, and momentum balances in the feed and permeate channels. This is the case with DCMD, where the process can be modeled using a coupled system of PDEs, in which the Navier–Stokes equations for the free fluid and the Darcy equations for the membrane are combined with the heat and transport equations.

According to the above, and in order to contribute to a better understanding of the desalination phenomena from a mathematical perspective, in this Doctoral Thesis, we study a conforming and non-conforming finite element methods to approximate the solution of a fluid-membrane model which arise from the equations present in a DCMD processes (see [87]). More precisely, we focus on analyzing numerical schemes for a simplified version of the equations modeling DCMD processes given by the coupling of the Navier–Stokes, Darcy and heat equations.

1.2 A brief description of the finite element method

Partial differential equations (PDE) arise in the mathematical modeling of many physical, chemical and biological phenomena and many diverse subject areas such as fluid dynamics, electromagnetism, material science, astrophysics, economy, financial modelling, among others. Very frequently the equations under consideration are so complicated that finding their analytical solutions (e.g. by Laplace and Fourier transform methods, or in the form of a power series) is either impossible or impracticable, and one has to resort to seeking numerical approximations to the unknown analytical solution.

Among the various methods available to approximate PDEs, we can mention Finite Difference Methods (FDM), Finite Volume Methods (FVM), Finite Element Methods (FEM), and Spectral Methods (SM), among others. Each of these methods has its own advantages and disadvantages.

However, our focus is on applying the Finite Element Method (FEM), introduced by Courant in 1943 in his work *"Variational methods for the solution of problems of equilibrium and vibrations"* and used to solve problems of solid and structural analysis (see [39]). To approximate the solutions of Partial Differential Equations using FEM, the following generic procedure is typically involved:

- First, a variational (weak) formulation of the problem must be considered. This allows the search for generalized solutions in Hilbert or Banach functional spaces.
- Then, projecting the solution into a finite-dimensional space allows calculating discrete approximations (the so-called Galerkin scheme) to the variational

problem.

- Finally, the approximation error of the method is determined by calculating the difference between the solution of the weak problem and the Galerkin problem.

Regarding Galerkin's method, it is flexible as it allows the use of different approaches to approximate the generalized solutions of a variational formulation. For instance, one can utilize Conforming Galerkin Methods (CG), Discontinuous Galerkin Methods (DG), Hybridizable Discontinuous Galerkin Methods (HDG), among others.

In particular, the Discontinuous Galerkin method was introduced by Reed and Hill in 1973 for a hyperbolic problem, in their work *"Triangular mesh methods for the neutron transport equation"* (see [92]), while the first analysis for a steady first-order PDEs was performed by Lesaint and Raviart in 1974 in the work *"On a finite element method for solving the neutron transport equation"* (see [75]).

Among the advantages of the DG method is its ability to construct finite-dimensional functional spaces without requiring continuity, making it well-suited for adaptability and parallel computing. However, this flexibility, which allows for discontinuities in finite element spaces, leads to a significant increase in computational expense compared to the Conforming Galerkin method.

To reduce the computational cost associated with DG methods, Cockburn *et. al.* (see [32]) introduced the Hybridizable Discontinuous Galerkin methods. The main idea of the HDG method is to introduce both a finite element space and an additional unknown at the interfaces between the mesh elements. In this way, this new trace unknown approximates the solution on the element boundaries and the numerical flux couples them indirectly through their communication with the approximate trace on the shared interface. Then, the weak continuity (in an integral sense) of the normal component of the numerical flux is enforced across the interfaces of mesh elements (transmission condition), and it allows for the elimination of the DOFs on the interior of each element using static condensation resulting in a global system of equations for only the single-valued trace unknown, and after that, the main unknown can be obtained independently inside each element.

Figure 1.4 illustrates that the continuity requirement of basis functions in the CG method enables shared degrees of freedom (DOFs) between neighboring elements. In contrast, DG methods do not share any DOFs between elements, leading to a duplication of DOFs and, consequently, an increase in the size of the global linear system. On the other hand, in HDG methods, instead of coupling the local solutions in neighboring mesh elements directly, the numerical flux couples them indirectly through its communication with the approximate trace at the shared interface.

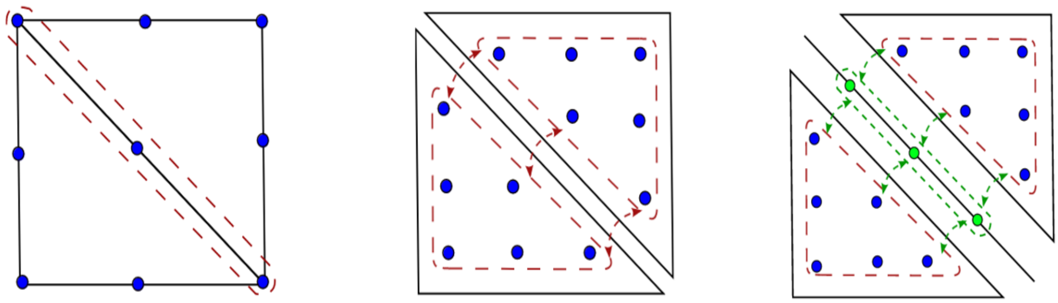

Figure 1.4: The coupling of DOFs on neighboring elements for the CG method (left), the DG method (center), and the HDG method (right).

1.3 Objectives and contributions

As previously mentioned in Section 1.1, the objective of this work is to develop conforming and non-conforming discretizations for a Navier–Stokes/Darcy/heat coupled system which describes several phenomena, in geophysics, oil extraction engineering and DCMD process for seawater desalinization, among others. More precisely, in this thesis we study numerical discretization for the mathematical model present later on in Chapter 2.

At this point, it is important to note that fluid-membrane interaction problems have been extensively studied in several works. Just to mention a few works in this direction, we can mention [5, 27, 28, 29, 45, 60, 61, 63] and their respective references, where several numerical methods are developed to approximate the solution of the Navier–Stokes/Darcy system, including conformal and non-conformal schemes. On the other hand, when it comes to numerical methods for coupling fluid flow to the heat equation, where the problem is the Navier–Stokes/heat coupling, commonly known as the Boussinesq problem, we can mention among others [9, 13, 25, 35, 37, 48, 49, 83, 84, 85] and their respective references.

In particular, in [63], the authors introduce and analyze a discontinuous Galerkin (DG) discretization for the nonlinear coupled problem, employing the nonsymmetric interior penalty Galerkin (NIPG), symmetric interior penalty Galerkin (SIPG), and incomplete interior penalty Galerkin (IIPG) bilinear forms for the discretization of the Laplacian in both media, and the upwind Lesaint–Raviart discretization of the convective term in the free fluid domain. On the other hand, in [5], the authors propose an iterative subdomain method that uses the velocity-pressure formulation for the Navier–Stokes equation and the primal formulation for the Darcy equation. Finally, in [45], the authors extend the work in [56] to the Navier–

Stokes/Darcy model and introduce a conforming numerical scheme to approximate the solution of the problem. The variational formulation is based on the standard velocity-pressure formulation for the Navier–Stokes equation and the dual-mixed formulation for the Darcy equation, resulting in the velocity and pressure of the fluid in both media as the main unknowns of the coupled system. Since one of the interface conditions becomes essential, they proceed similarly to [56] and incorporate the trace of the porous medium pressure as an additional unknown.

Despite the above, the number of contributions for the Darcy-heat coupled system is relatively limited. In fact, the first contribution on the analysis of a finite element method for Darcy's problem coupled with the heat equation is presented in [7] (see also [8]). There, the authors introduce two finite element discretizations for the coupled system with temperature-dependent viscosity. One of the difficulties in the analysis of [7] is the fact that the velocity lives in H(div), which forces the trial and test spaces for the temperature to be different, preventing the utilization of classical results for elliptic problems to obtain the well-posedness of the continuous formulation. More recently, in [57] it is introduced a new fully-mixed finite element method for the model studied in [7]. There, the authors employ a Banach spaces-based analysis to prove well-posedness of the continuous problem and its corresponding finite element discretization.

However, to the best of our knowledge, this thesis constitutes the first attempt in the literature to carry out an analysis for a nonisothermal fluid-membrane phenomena.

The main contributions of this thesis are:

- Proposing a non-isothermal fluid-membrane interaction model.
- Introducing a variational formulation and analyzing the existence and uniqueness of the solution.
- Establishing a conforming Galerkin scheme, analyzing the solvability of the scheme, and proving the corresponding stability and convergence results.
- Presenting a mass-conservative variational formulation and analyzing the existence and uniqueness of the solution.
- Constructing a nonconforming Galerkin scheme, analyzing the solvability of the scheme, and proving the corresponding stability and convergence results.
- Developing an a posteriori error analysis for the conformal Galerkin scheme.
- Implementing numerical simulations to validate the theoretical results.

1.4 Thesis outline

The remaining chapters of this thesis are structured as follows:

In **Chapter 2**, we introduce the model problem for nonisothermal fluid-membrane interaction. The problem involves two different regions: one occupied by the fluid and the other by the porous medium. The governing equations consist of the Navier–Stokes/heat system in the free-fluid region and a Darcy-heat coupled system in the membrane, along with the appropriate interface and boundary conditions.

In **Chapter 3**, we propose and analyze a conforming finite element method for the nonisothermal fluid-membrane interaction problem introduced in Chapter 2. We consider the standard velocity-pressure-temperature variational formulation for the Navier–Stokes/heat system in the free-fluid region, along with a dual-mixed scheme coupled with a primal formulation for the Darcy-heat system in the membrane region. The latter yields the introduction of the trace of the porous medium pressure as a suitable Lagrange multiplier. For the associated Galerkin scheme, we employ Bernardi–Raugel and Raviart–Thomas elements for velocities, piecewise constant elements for pressures, continuous piecewise linear functions for temperatures, and continuous piecewise linear functions for the Lagrange multiplier on a partition of the interface. We prove well-posedness for both the continuous and discrete schemes and derive corresponding error estimates. The contents of this chapter gave rise to the following paper:

[20] J. Camaño, R. Oyarzúa, M. Serón, and M. Solano. *A conforming finite element method for a nonisothermal fluid-membrane interaction.* Math. Comp., In press, 2025.

In **Chapter 4**, we propose a residual-based a posteriori error estimator for the conforming discretization of a nonisothermal Navier–Stokes/Darcy coupled system introduced in Chapter 3. More precisely, we introduce a reliable and efficient a posteriori error estimator for the aforementioned coupled system. Employing standard arguments such as global inf-sup conditions, suitable Helmholtz's decompositions and the local approximation properties of the Raviart–Thomas and Clément interpolation operators, we derive the a posteriori error estimator and prove its reliability. In turn, inverse inequalities, usual localization techniques of bubble functions and known results from previous works, are employed to prove the local efficiency of the proposed error estimator. Finally, we propose an adaptive algorithm based on the local and computable error indicators of the a posteriori error estimator.

The contents of this chapter are part of a current work.

In **Chapter 5**, we propose and analyze an $\mathbf{H}(\mathrm{div})$-conforming and mass conservative finite element method for the nonisothermal fluid-membrane interaction problem introduced in Chapter 2. We consider a velocity-pressure-temperature variational scheme for the Navier–Stokes/heat system in the free-fluid region whereas in the membrane region we consider a dual-mixed formulation for the Darcy system coupled with a primal equation for the temperature model. In this way, the unknowns of the resulting formulation are given by the velocity, pressure and temperature in both domains. For the associated Galerkin scheme, we combine an $\mathbf{H}(\mathrm{div})$-conforming scheme for the fluid variables and a conforming Galerkin discretization for the heat equation. Therefore, the resulting numerical scheme produces exactly divergence-free velocities and also allows preserve the law of conservation of mass at a discrete level. The contents of this chapter gave rise to the following papers:

[21] J. Camaño, R. Oyarzúa, M. Serón, and M. Solano. *A strong mass conservative finite element method for the Navier–Stokes/Darcy coupled system.* Appl. Math. Lett, 163: 109447, 2025.

and

[19] J. Camaño, R. Oyarzúa, M. Serón, and M. Solano. *A mass conservative finite element method for a nonisothermal Navier–Stokes/Darcy coupled system.* Preprint on webpage at `https://www.ci2ma.udec.cl/publicaciones/prepublicaciones/prepublicacion.php?id=536`

1.5 Preliminary notations

In this section, we present some definitions and notations that we will use throughout the rest of the present Thesis.

Let $\mathcal{O} \subseteq \mathbb{R}^d$, $d \in \{2, 3\}$, denote a domain with Lipschitz boundary Γ. For $s \geq 0$ and $p \in [1, +\infty]$, we denote by $\mathrm{W}^{s,p}(\mathcal{O})$ the usual Sobolev space endowed with the norm $\|\cdot\|_{s,p,\mathcal{O}}$. If $s = 0$, $\mathrm{W}^{0,p}(\mathcal{O})$ corresponds to the usual Lebesgue space $\mathrm{L}^p(\mathcal{O})$, which is endowed with the norm $\|\cdot\|_{0,p,\mathcal{O}}$. If $p = 2$, we write $\mathrm{H}^s(\mathcal{O})$ in place of $\mathrm{W}^{s,2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$, respectively, and the seminorm by $|\cdot|_{s,\mathcal{O}}$. In addition, $\mathrm{H}^1_0(\mathcal{O})$ will denote the space of functions in $\mathrm{H}^1(\mathcal{O})$ with null trace on Γ, and $\mathrm{L}^2_0(\mathcal{O})$ will be the space of $\mathrm{L}^2(\mathcal{O})$ functions with zero mean value over $\mathcal{O}$, that is

$$\mathrm{L}^2_0(\mathcal{O}) := \left\{ v \in \mathrm{L}^2(\mathcal{O}) : \int_{\mathcal{O}} v = 0 \right\}.$$

Given $p, q \in (1, +\infty)$ satisfying $1/p + 1/q = 1$, in what follows, we will denote by $\mathrm{W}^{1/q,p}(\Gamma)$ the trace space of $\mathrm{W}^{1,p}(\mathcal{O})$ and by $\mathrm{W}^{-1/q,q}(\Gamma)$ the dual space of $\mathrm{W}^{1/q,p}(\Gamma)$ endowed with the norms $\|\cdot\|_{1/q,p,\Gamma}$ and $\|\cdot\|_{-1/q,q,\Gamma}$, defined respectively by

$$\|\phi\|_{1/q,p,\Gamma} := \inf\left\{\|\psi\|_{1,p,\mathcal{O}} : \psi \in \mathrm{W}^{1,p}(\mathcal{O}), \psi|_\Gamma = \phi\right\} \qquad \forall \phi \in \mathrm{W}^{1/q,p}(\Gamma),$$

and

$$\|\psi\|_{-1/q,p,\Gamma} = \sup_{\xi \in \mathrm{W}^{1/q,p}(\Gamma)\setminus\{\mathbf{0}\}} \frac{\langle \psi, \xi\rangle_\Gamma}{\|\xi\|_{1/q,p,\Gamma}} \qquad \forall \psi \in \mathrm{W}^{-1/q,q}(\Gamma).$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality parity between $\mathrm{W}^{-1/q,q}(\Gamma)$ and $\mathrm{W}^{1/q,p}(\Gamma)$, which coincides with the inner product on $\mathrm{L}^2(\Gamma)$ when restricted to $\mathrm{L}^2(\Gamma)$. When $p = 2$, we will write $\mathrm{H}^{1/2}(\Gamma) := \mathrm{W}^{1/2,2}(\Gamma)$, $\|\cdot\|_{1/2,2,\Gamma} = \|\cdot\|_{1/2,\Gamma}$, $\mathrm{H}^{-1/2}(\Gamma) := \mathrm{W}^{-1/2,2}(\Gamma)$ and $\|\cdot\|_{-1/2,2,\Gamma} = \|\cdot\|_{-1/2,\Gamma}$. In addition, we denote by $\mathbf{M}$ and $\mathbb{M}$ the corresponding vectorial and tensorial counterparts of the generic scalar functional space M.

Additionally, we recall that

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \Big\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \ \mathrm{div}\,\mathbf{w} \in \mathrm{L}^2(\mathcal{O})\Big\},$$

endowed with the norm $\|\mathbf{w}\|_{\mathrm{div},\mathcal{O}} := \left(\|\mathbf{w}\|_{0,\mathcal{O}}^2 + \|\mathrm{div}\,\mathbf{w}\|_{0,\mathcal{O}}^2\right)^{1/2}$ is a standard Hilbert space in the realm of mixed problems (see, e.g., [18]).

For simplicity, in what follows for any scalar fields v and w, vector fields $\mathbf{v} = (v_i)_{i=1,d}$ and $\mathbf{w} = (w_i)_{i=1,d}$, and tensor fields $\mathbf{A} = (a_{ij})_{i,j=1,d}$ and $\mathbf{B} = (b_{ij})_{i,j=1,d}$, we will denote

$$(v, w)_\mathcal{O} := \int_\mathcal{O} vw, \quad (\mathbf{v}, \mathbf{w})_\mathcal{O} := \int_\mathcal{O} \mathbf{v}\cdot\mathbf{w}, \quad \text{and} \quad (\mathbf{A}, \mathbf{B})_\mathcal{O} := \int_\mathcal{O} \mathbf{A} : \mathbf{B},$$

where $\mathbf{A} : \mathbf{B} := \sum_{i,j=1}^{d} a_{ij} b_{ij}$.

In turn, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$. Furthermore, given a non-negative integer k and a subset S of $\mathbb{R}^d$, $\mathrm{P}_k(S)$ stands for the space of polynomials defined on S of degree equal or less than k.

Moreover, throughout the rest of this Thesis, we will employ $\mathbf{0}$ as a generic null vector (or tensor), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

1.6 Preliminary results

We now recall some results from functional and numerical analysis that will be used in the analysis developed in the following chapters.

Theorem 1.6.1. *(Hölder's inequality [30, Theorem 2.5-1(a)], [47, Theorem B.6], or [16, Theorem 4.6]). Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$. Given a real number $p > 1$, let the real number q be defined by $\frac{1}{p} + \frac{1}{q} = 1$, and let $f \in \mathrm{L}^p(\mathcal{O})$ and $g \in \mathrm{L}^q(\mathcal{O})$. Then $fg \in \mathrm{L}^1(\mathcal{O})$ and*

$$|(f, g)_{\mathcal{O}}| \leq \|f\|_{0,p,\mathcal{O}} \|g\|_{0,q,\mathcal{O}}. \tag{1.1}$$

Remark 1.6.1. *(Generalized Hölder's inequality [16, Chapter 4, Remark 2]) Hölder's inequality can be extended as follows: Assume that $f_1, f_2, \ldots, f_k$ are functions such that $f_i \in \mathrm{L}^{p_i}(\mathcal{O})$, $1 \leq i \leq k$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \leq 1$. Then the product $f = f_1 f_2 \cdots f_k$ belongs to $\mathrm{L}^p(\mathcal{O})$ and*

$$\|f\|_{0,p,\mathcal{O}} \leq \|f_1\|_{0,p_1,\mathcal{O}} \|f_2\|_{0,p_2,\mathcal{O}} \cdots \|f_k\|_{0,p_k,\mathcal{O}}. \tag{1.2}$$

Theorem 1.6.2. *(Poincaré's inequality [30, Theorem 6.5-2(a)] or [47, Lemma B.61]). Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$. For each $1 \leq p < \infty$, there exists a positive constant $c = c(\mathcal{O}, p)$ such that*

$$\|v\|_{0,p,\mathcal{O}} \leq c|v|_{1,p,\mathcal{O}} \qquad \forall\, v \in \mathrm{W}_0^{1,p}(\mathcal{O}). \tag{1.3}$$

Theorem 1.6.3. *(Korn's inequality [30, Theorem 6.15-1]). Let $\mathcal{O}$ be a domain in $\mathbb{R}^d$. There exists a constant $c = c(\mathcal{O})$ such that*

$$\|\mathbf{v}\|_{1,\mathcal{O}} \leq c(\|\mathbf{v}\|_{0,\mathcal{O}}^2 + \|\mathbf{e}(\mathbf{v})\|_{0,\mathcal{O}}^2)^{1/2} \qquad \forall\, \mathbf{v} \in \mathbf{H}^1(\mathcal{O}), \tag{1.4}$$

where

$$\mathbf{e}(\mathbf{v}) := (e_{ij}(\mathbf{v})) \quad \textit{with} \quad e_{ij}(\mathbf{v}) := \tfrac{1}{2}(\partial_j v_i + \partial_i v_j), 1 \leq i, j \leq d.$$

Theorem 1.6.4. *(Trace inequality [52, Theorem 1.4]). Let $\mathcal{O}$ be a bounded domain of $\mathbb{R}^d$ with Lipschitz-continuous boundary Γ, and let $\gamma_0 : D(\mathcal{O}) \to \mathrm{L}^2(\Gamma)$ be the mapping defined by*

$$\gamma_0(\varphi) := \varphi|_\Gamma \qquad \forall\, \varphi \in D(\overline{\mathcal{O}}).$$

Then there exists $C > 0$ such that

$$\|\gamma_0(\varphi)\|_{0,\Gamma} \leq c\|\varphi\|_{1,\mathcal{O}} \qquad \forall\, \varphi \in D(\overline{\mathcal{O}}). \tag{1.5}$$

Theorem 1.6.5. *(Sobolev embedding [30, Theorem 6.6-3] (or [89, Theorem 1.3.4] or [47, Corollary B.43.]) and [15, Proposition 1.4.2.]). For all $q > \frac{d}{r}$, and for all $1 \leq t \leq s \leq \infty$, there hold*

$$\mathrm{W}^{q,r}(\mathcal{O}) \hookrightarrow \mathrm{C}^0(\overline{\mathcal{O}}) \quad \textit{and} \quad \mathrm{W}^{1,s}(\mathcal{O}) \hookrightarrow \mathrm{W}^{1,t}(\mathcal{O}). \tag{1.6}$$

Theorem 1.6.6. *(Sobolev inequality [47, Theorem B.40]). For $1 \leq q < \infty$ when $d = 2$ and $1 \leq q \leq 6$ when $d = 3$, $\mathrm{H}^1(\mathcal{O})$ is continuously embedded into $\mathrm{L}^q(\mathcal{O})$. In addition, there exists a constant $c > 0$, depending only on $|\mathcal{O}|$ and q, such that*

$$\|w\|_{0,q,\mathcal{O}} \leq c\|w\|_{1,\mathcal{O}} \qquad \forall\, w \in \mathrm{H}^1(\mathcal{O}). \tag{1.7}$$

Theorem 1.6.7. *(Lax–Milgram Lemma [30, Theorem 6.2-1], [47, Lemma 2.2], or [52, Theorem 1.1]). Let V be a Hilbert space, let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a continuous and V-coercive bilinear form, and let $f : V \to \mathbb{R}$ be a continuous linear functional. In other words, there exist constants $c_1, c_2, c_3 > 0$ such that for all $u, v \in V$*

$$|a(u,v)| \leq c_1 \|u\|_V \|v\|_V, \qquad |a(v,v)| \geq c_2 \|v\|_V^2, \qquad |\langle f, v\rangle| \leq c_3 \|v\|_V.$$

Then, the following variational problem: Find $u \in V$ such that

$$a(u,v) = f(v) \qquad \forall\, v \in V,$$

has one and only one solution.

Theorem 1.6.8. *(Babuška–Brezzi's Theorem [30, Theorem 6.12-1]). Let V and Q be two Hilbert spaces, and let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and $b(\cdot,\cdot) : V \times Q \to \mathbb{R}$ be two continuous bilinear forms with the following properties:*

1. *There exists a constant $\alpha > 0$ such that*

$$a(v,v) \geq \alpha \|v\|_V^2 \quad \text{for all} \quad v \in V_0 := \{v \in V : b(v,q) = 0 \quad \forall\, q \in Q\},$$

2. *There exists a constant $\beta > 0$ such that*

$$\sup_{\substack{v \in V \\ v \neq 0}} \frac{|b(v,q)|}{\|v\|_V} \geq \beta \|q\|_Q \qquad \forall q \in Q.$$

Finally, let $f : V \to \mathbb{R}$ and $g : Q \to \mathbb{R}$ be two continuous linear functionals. Then, the following abstract variational problem: Find $(u,p) \in V \times Q$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v\rangle_{V' \times V} \qquad && \forall\, v \in V, \\ b(u,q) &= \langle g, q\rangle_{Q' \times Q} \qquad && \forall\, q \in Q, \end{aligned}$$

has one and only one solution.

Theorem 1.6.9. *(Brouwer fixed-point Theorem [30, Theorem 9.9-2]). Let K be a compact and convex subset of a finite dimensional Banach space X, and let $f : K \to K$ be a continuous mapping. Then f has at least one fixed point.*

Theorem 1.6.10. *(Banach fixed-point Theorem [30, Theorem 3.7-1]). Let X be a complete metric space. Then any contraction $f : X \to X$ has one and only one fixed point $x \in X$.*

Theorem 1.6.11. *([30, Theorem 6.5-1] or [16, Proposition 9.1]). Let $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ and let $m \geq 1$ be an integer. The Sobolev space $\mathrm{W}^{m,p}(\mathcal{O})$ is a Banach space, separable if $1 \leq p < \infty$, and reflexive if $1 < p < \infty$.*

Definition 1.6.1. *Let X be a normed space and let X' denote its dual. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to convergence weakly in X if there exists $x \in X$ such that*

$$x'(x_n) \overset{n\to\infty}{\to} x'(x) \qquad \text{for each } x' \in X'.$$

Here, x is called a weak limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$ and the weak convergence is denoted as

$$x_n \overset{n\to\infty}{\rightharpoonup} x.$$

Theorem 1.6.12. *([30, Theorems 5.12-1 and 5.12-2]).*

1. *In any normed vector space $x_n \to x$ as $n \to \infty$ implies that $x_n \rightharpoonup x$ as $n \to \infty$.*
2. *In a finite-dimensional normed vector space, any weakly convergent sequence is also strongly convergent.*
3. *The limit of a weakly convergent sequence is unique.*
4. *A weakly convergent sequence is bounded.*
5. *Let $x_n \rightharpoonup x$ as $n \to \infty$. Then*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Theorem 1.6.13. *([30, Theorem 5.12-4]). Let X and Y be normed vector spaces over the same field $\mathbb{K}$. Additionally, let $A : X \to Y$ be a linear operator and $B : X \times Y \to \mathbb{K}$ be a bilinear operator.*

1. $x_n \overset{n\to\infty}{\rightharpoonup} x$ *in* X *implies* $Ax_n \overset{n\to\infty}{\rightharpoonup} Ax$ *in* Y.
2. *Let A be a compact operator.* $x_n \overset{n\to\infty}{\rightharpoonup} x$ *in* X *implies* $Ax_n \overset{n\to\infty}{\to} Ax$ *in* Y.
3. $x_n \overset{n\to\infty}{\rightharpoonup} x$ *in* X *and* $y_n \overset{n\to\infty}{\rightharpoonup} y$ *in* X *implies* $B(x_n, y_n) \overset{n\to\infty}{\to} B(x, y)$ *in* $\mathbb{K}$.

Theorem 1.6.14. *([16, Theorem 3.18]) Let X be a reflexive Banach space and let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in X. Then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges weakly to $x \in X$.*

Theorem 1.6.15. *([16, Proposition 3.20]) Let X be a reflexive Banach space and let $Y \subseteq X$ be a closed linear subspace of X. Then Y is reflexive.*

Chapter 2

The model problem

In this chapter, we introduce a nonisothermal fluid-membrane interaction problem that arises in membrane desalination processes [88, 101]. The problem consists of a Navier–Stokes/heat system, commonly known as the Boussinesq system, in the free-fluid region, and a Darcy-heat coupled system in the membrane. These systems are coupled through buoyancy terms and a set of transmission conditions on the fluid-membrane interface, including mass conservation, balance of normal forces, the Beavers–Joseph–Saffman law, and continuity of heat flux and fluid temperature. Finally, we complete the definition for the coupled problem introducing suitable boundary conditions.

2.1 The geometry configuration

In order to describe the geometry of the problem, we let Ω_{f} and Ω_{m} be two bounded and simply connected open polygonal or polyhedral domains in $\mathbb{R}^d$, with $d = 2$ or $d = 3$, respectively, such that $\Omega_{\mathrm{f}} \cap \Omega_{\mathrm{m}} = \emptyset$ and $\Sigma := \partial\Omega_{\mathrm{f}} \cap \partial\Omega_{\mathrm{m}} \neq \emptyset$ is either the union of straight lines if $d = 2$ (see Fig. 2.1) or a polygonal surface if $d = 3$, satisfying $\overline{\Sigma} \subsetneq \partial\Omega_{\mathrm{f}}$ and $\overline{\Sigma} \subsetneq \partial\Omega_{\mathrm{m}}$. Then, we let $\Gamma_{\mathrm{f}} := \partial\Omega_{\mathrm{f}} \setminus \overline{\Sigma}$, $\Gamma_{\mathrm{m}} := \partial\Omega_{\mathrm{m}} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_{\mathrm{f}} \cup \Omega_{\mathrm{m}} \cup \Sigma$ and Ω_{f} (and hence inward to Ω_{m} when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t} = \mathbf{t}_1$ when $d = 2$ (see Fig. 2.1) and by $\mathbf{t}_1, \mathbf{t}_2$, when $d = 3$.

The problem we are interested in consists of the movement of an incompressible viscous fluid subject to a heat source occupying Ω_{f} which flows towards and from a porous membrane Ω_{m} through Σ, where Ω_{m} is saturated with the same fluid (see [79, 88]). The mathematical model is defined by two separate groups of equations, a set of coupling terms and a set of boundary conditions.

This domain configuration can be found in various problems involving the

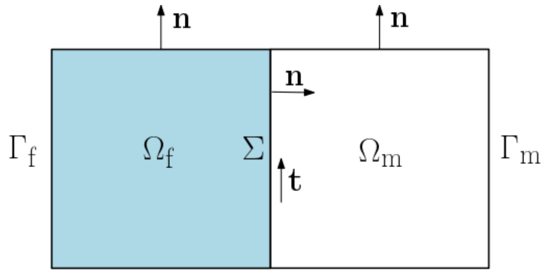

Figure 2.1: Sketch of a 2D geometry for the coupled model.

filtration of incompressible fluids through porous media (see for example [44, 86]).

2.2 Navier–Stokes/heat system

The Navier–Stokes equations provide a model for the flow motion of a homogeneous, incompressible Newtonian fluid (a rigorous derivation of the Navier–Stokes equations can be found in [50]). In particular, we adopt the mesoscopic model given in [86, Section 3.3.1]. Furthermore, we couple the nonlinear Navier–Stokes system with the heat equation. This coupling is usually known as the Boussinesq model, in honor to the french mathematician and physicist Joseph V. Boussinesq, who proposed and studied this phenomena in 1897 [14].

According to the above, in the free fluid domain Ω_{f}, the motion of the fluid can be described by the following Navier–Stokes/heat system:

$$\boldsymbol{\sigma}_{\mathrm{f}} = 2\mu\,\mathbf{e}(\mathbf{u}_{\mathrm{f}}) - p_{\mathrm{f}}\,\mathbf{I} \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{2.1a}$$

$$-\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{f}} + (\mathbf{u}_{\mathrm{f}}\cdot\nabla)\,\mathbf{u}_{\mathrm{f}} - \mathbf{g}_{\mathrm{f}}\theta_{\mathrm{f}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{2.1b}$$

$$\operatorname{div}\mathbf{u}_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{2.1c}$$

$$-\kappa_{\mathrm{f}}\,\Delta\,\theta_{\mathrm{f}} + \mathbf{u}_{\mathrm{f}}\cdot\nabla\,\theta_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{2.1d}$$

where $\mu > 0$ is the dynamic viscosity of the fluid, $\mathbf{u}_{\mathrm{f}}$ is the fluid velocity, p_{f} is the fluid pressure, $\boldsymbol{\sigma}_{\mathrm{f}}$ is the Cauchy stress tensor, $\mathbf{I}$ is the $d\times d$ identity matrix, θ_{f} is the fluid temperature, $\kappa_{\mathrm{f}} > 0$ is the fluid thermal conductivity, $\mathbf{g}_{\mathrm{f}} \in \mathbf{L}^2(\Omega_{\mathrm{f}})$ is the external force per unit mass, $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor, and $\mathbf{e}(\mathbf{u}_{\mathrm{f}})$ is the strain rate tensor given by

$$\mathbf{e}(\mathbf{u}_{\mathrm{f}}) := \frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{f}} + (\nabla\mathbf{u}_{\mathrm{f}})^t\right),$$

where the superscript t denotes transposition.

2.3 Darcy-heat system

The filtration of an incompressible fluid through porous media is often described using Darcy's law, which is a linear relationship between the velocity and pressure of an incompressible fluid in a saturated porous medium. To establish the governing equations in this domain, we adopt the mesoscopic model given in [86, Section 3.3.1] for Darcy's law, and similarly to the above, we couple this system with the convection-diffusion equation. In this way, in the porous membrane Ω_{m} we have the following Darcy-heat system,

$$\mathbf{K}^{-1}\,\mathbf{u}_{\mathrm{m}} + \nabla p_{\mathrm{m}} - \mathbf{g}_{\mathrm{m}}\theta_{\mathrm{m}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{2.2a}$$

$$\operatorname{div}\mathbf{u}_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{2.2b}$$

$$-\kappa_{\mathrm{m}}\,\Delta\,\theta_{\mathrm{m}} + \mathbf{u}_{\mathrm{m}}\cdot\nabla\,\theta_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{2.2c}$$

where $\mathbf{u}_{\mathrm{m}}$ represents the fluid velocity, p_{m} the fluid pressure, θ_{m} the fluid temperature, $\mathbf{g}_{\mathrm{m}} \in \mathbf{L}^3(\Omega_{\mathrm{m}})$ a given external force, $\kappa_{\mathrm{m}} > 0$ the thermal conductivity, and $\mathbf{K} \in \mathbb{L}^\infty(\Omega_{\mathrm{m}})$ is a symmetric and uniformly positive definite tensor in Ω_{m} representing the intrinsic permeability $\boldsymbol{\kappa}$ of the membrane divided by the dynamic viscosity μ of the fluid. Throughout the thesis we assume that there exists $C_{\mathbf{K}} > 0$ such that

$$\boldsymbol{\xi}^t\mathbf{K}(x)\,\boldsymbol{\xi} \;\geq\; C_{\mathbf{K}}\,|\boldsymbol{\xi}|^2, \tag{2.3}$$

for almost all $x \in \Omega_{\mathrm{m}}$, and for all $\boldsymbol{\xi} \in \mathbb{R}^d$.

2.4 The transmission conditions

We now focus on establishing coupling conditions across the interface Σ separating the free fluid domain and the porous medium. This is a classic problem that has been investigated from both a rigorous physical and mathematical point of view since one of the mathematical difficulties arises from the fact that we need to couple two different systems of partial differential equations. The conditions that must be prescribed on Σ are the following:

1. The condition to assign at a permeable interface is the continuity of the normal velocity, which is a consequence of the incompressibility of the fluid.

2. A condition relating the pressures of the two fluids across the interface Σ.

3. A condition on the tangential component for the fluid velocity at the interface Σ.

4. A condition relating the temperatures of the two domains across the interface Σ.

5. A condition that relates the thermal conductivities of the two domains through the Σ interface.

Therefore, the transmission conditions that couple the systems (2.1) and (2.2) on the interface Σ are given by

$$\theta_{\mathrm{f}} = \theta_{\mathrm{m}} \quad \text{on} \quad \Sigma, \tag{2.4a}$$

$$\kappa_{\mathrm{f}} \nabla \theta_{\mathrm{f}} \cdot \mathbf{n} = \kappa_{\mathrm{m}} \nabla \theta_{\mathrm{m}} \cdot \mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.4b}$$

$$\mathbf{u}_{\mathrm{f}} \cdot \mathbf{n} = \mathbf{u}_{\mathrm{m}} \cdot \mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.4c}$$

$$\boldsymbol{\sigma}_{\mathrm{f}} \, \mathbf{n} + \sum_{j=1}^{d-1} \omega_j (\mathbf{u}_{\mathrm{f}} \cdot \mathbf{t}_j) \, \mathbf{t}_j = -p_{\mathrm{m}} \, \mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.4d}$$

where $\omega_1, \ldots, \omega_{d-1}$ are positive constants depending on the intrinsic permeability $\boldsymbol{\kappa}$, the viscosity μ, and on the geometrical characteristics of the membrane (see [6]).

The first and second conditions represent the continuity of the temperature and the heat flux, respectively, while (2.4c) is a consequence of the incompressibility of the fluid and the conservation of mass across Σ (see [79]). In turn, the fourth condition (2.4d) can be decomposed, at least formally, into its normal and tangential components as follows:

$$\begin{aligned} (\boldsymbol{\sigma}_{\mathrm{f}} \, \mathbf{n}) \cdot \mathbf{n} &= -p_{\mathrm{m}} \\ (\boldsymbol{\sigma}_{\mathrm{f}} \, \mathbf{n}) \cdot \mathbf{t}_j &= -\omega_j (\mathbf{u}_{\mathrm{f}} \cdot \mathbf{t}_j) \quad \text{on} \quad \Sigma, \quad j \in 1, \ldots, d-1. \end{aligned} \tag{2.5}$$

The first equation in (2.5) corresponds to the balance of normal forces, whereas the second one is known as the Beavers–Joseph–Saffman law, which establishes that the slip velocity along Σ is proportional to the shear stress along Σ (assuming also, based on experimental evidence, that $\mathbf{u}_{\mathrm{m}} \cdot \mathbf{t}$ is negligible). We refer to [6, 81, 94] for further details on this interface condition.

2.5 The boundary conditions

In order to complete the definition of our coupled problem, we have to introduce suitable boundary conditions.

1. For the Navier–Stokes equations, we consider a no-slip condition on the boundary Γ_{f}, that is, the velocity vanishes on the boundary.

2. For the Darcy problem, we consider a no-flow condition on the boundary Γ_{m}, that is, the normal component of the velocity vanishes on the boundary.

3. For the Heat problem, we consider a set temperature boundary on boundaries Γ_{f} and Γ_{m} described by a Dirichlet's type boundary.

Therefore, the boundary conditions for the Navier–Stokes/Darcy/heat coupled system (2.1), (2.2) and (2.4), are given by:

$$\begin{aligned}
\mathbf{u}_{\mathrm{f}} &= \mathbf{0} && \text{on} \quad \Gamma_{\mathrm{f}}, \\
\mathbf{u}_{\mathrm{m}} \cdot \mathbf{n} &= 0 && \text{on} \quad \Gamma_{\mathrm{m}}, \\
\theta_{\mathrm{f}} &= \theta_{\mathrm{D}}|_{\Gamma_{\mathrm{f}}} && \text{on} \quad \Gamma_{\mathrm{f}}, \\
\theta_{\mathrm{m}} &= \theta_{\mathrm{D}}|_{\Gamma_{\mathrm{m}}} && \text{on} \quad \Gamma_{\mathrm{m}},
\end{aligned} \tag{2.6}$$

where $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ is given function defined on $\Gamma := \Gamma_{\mathrm{f}} \cup \Gamma_{\mathrm{m}}$.

To conclude this chapter, it is important to note that in subsequent chapters, the proof of existence and stability of solutions to the continuous problem relies on introducing a suitable lifting for the Dirichlet datum θ_{D}. This lifting, as we will clarify in our subsequent analysis, must belong to $\mathrm{L}^{\infty}(\Omega)$. However, and as indicated by the first embedding in (1.6), this requirement is met if the lifting is in $\mathrm{W}^{1,r}(\Omega)$, with $r > 3$ for the three-dimensional scenario and $r > 2$ in two dimensions, suggesting that θ_{D} should be an element of $\mathrm{W}^{1-1/r,r}(\Gamma)$. Therefore, for the sake of simplifying both the analysis and notations, we set $r = 4$ henceforth and assume that $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$, ensuring the necessary condition for the lifting in both two and three dimensions.

Chapter 3

A conforming finite element method for a nonisothermal fluid-membrane interaction

3.1 Introduction

In this chapter, we focus on proposing and analyzing a conforming numerical scheme for the nonisothermal fluid-membrane model introduced in Chapter 2.

We use a velocity-pressure-temperature variational formulation in both domains, which yields the introduction of a suitable Lagrange multiplier representing the trace of the porous media pressure on the interface. Then, we combine the theory developed in [45] and [7], and similarly to [37] (see also [9]), make use of a suitable lifting of the temperature data to prove the existence of a solution by means of a fixed-point strategy and under a smallness assumption on data. In addition, under a restrictive assumption on the temperature solution, we prove uniqueness.

In terms of the discretization of the formulation, we utilize Bernardi–Raugel and Raviart–Thomas elements for the velocities in the free-fluid and porous media domains, respectively. For the pressure and temperature, we employ piece-wise constant and Lagrange elements, respectively, in both domains. Additionally, for the Lagrange multiplier, we use continuous and piece-wise linear functions. The analysis of the discrete scheme employs a similar approach to the continuous case, and under a smallness assumption on data and on the temperature in the membrane, we obtain the corresponding convergence rate of the Galerkin scheme and we will finish with some numerical implementations that corroborate the theoretical analysis.

The remainder of this chapter is organized as follows:

- In **Section 3.2**, we derive the corresponding variational formulation and analyze the existence and uniqueness of the solution.

- In **Section 3.3**, we define a conforming numerical scheme and analyze its well-posedness.

- In **Section 3.4**, we perform the error analysis and derive the corresponding order of convergence of the scheme.

- In **Section 3.5**, we provide some numerical results illustrating the performance of the method and confirming the theoretical rate of convergence.

3.2 Continuous problem

In this section, we derive the corresponding variational formulation for the model problem presented in Chapter 2. Next, we discuss the stability properties of the different forms involved and analyze the well-posedness of the resulting formulation.

3.2.1 The variational formulation

To derive a weak formulation for the coupled problem given by (2.1), (2.2), (2.4) and (2.6), we proceed similarly to [56, Section 2]. First, we introduce some more notation and definitions. In the sequel we will employ the following subspaces of $\mathbf{H}(\mathrm{div};\Omega_{\mathrm{m}})$ and $\mathrm{H}^1(\Omega_\star)$, respectively

$$
\begin{aligned}
\mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}}) \quad &:= \Big\{ \mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{m}}) : \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{m}} \Big\}, \\
\mathrm{H}^1_{\Gamma_\star}(\Omega_\star) \quad &:= \Big\{ \mathrm{v} \in \mathrm{H}^1(\Omega_\star) : \quad \mathrm{v} = 0 \quad \text{on} \quad \Gamma_\star \Big\},
\end{aligned}
$$

with $\star \in \{\mathrm{f},\mathrm{m}\}$. Notice that the latter implies the definition of the following subspace of $\mathbf{H}^1(\Omega_{\mathrm{f}})$

$$
\mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) := [\mathrm{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})]^d.
$$

To derive our weak formulation, first we multiply (2.1b) by a test function $\mathbf{v}_{\mathrm{f}} \in \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$, integrate by parts and employ (2.1a) and (2.4d), to obtain

$$
\begin{aligned}
2\mu\,(\mathbf{e}(\mathbf{u}_{\mathrm{f}}),\mathbf{e}(\mathbf{v}_{\mathrm{f}}))_{\Omega_{\mathrm{f}}} \,+\, \sum_{j=1}^{d-1} \omega_j \,\langle \mathbf{u}_{\mathrm{f}}\cdot\mathbf{t}_j, \mathbf{v}_{\mathrm{f}}\cdot\mathbf{t}_j\rangle_\Sigma \,+\, ((\mathbf{u}_{\mathrm{f}}\cdot\nabla)\,\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} \\
-\,(p_{\mathrm{f}},\mathrm{div}\,\mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} \,+\, \langle\lambda,\mathbf{v}_{\mathrm{f}}\cdot\mathbf{n}\rangle_\Sigma \,-\, (\theta_{\mathrm{f}}\,\mathbf{g}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} \,=\, 0,
\end{aligned}
\tag{3.1a}
$$

for all $\mathbf{v}_{\mathrm{f}} \in \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$, where $\lambda \in \mathrm{H}^{1/2}(\Sigma)$ is a further unknown representing the trace of the porous medium pressure on Σ, that is $\lambda = p_{\mathrm{m}}|_\Sigma$.

Next, we multiply (2.2a) by $\mathbf{v}_{\mathrm{m}} \in \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}})$, and integrate by parts to obtain

$$(\mathbf{K}_{\mathrm{m}}^{-1}\mathbf{u}_{\mathrm{m}}, \mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}} - \langle \mathbf{v}_{\mathrm{m}} \cdot \mathbf{n}, \lambda \rangle_\Sigma - (p_{\mathrm{m}}, \mathrm{div}\,\mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}} - (\theta_{\mathrm{m}}\,\mathbf{g}_{\mathrm{m}}, \mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}} = 0, \quad (3.1b)$$

for all $\mathbf{v}_{\mathrm{m}} \in \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}})$.

Now, to incorporate (2.1d) and (2.2c) to the variational system, we first define the spaces

$$\Psi_\infty := \{\psi \in \mathrm{H}^1(\Omega) \,:\, \psi|_{\Omega_{\mathrm{m}}} \in \mathrm{L}^\infty(\Omega_{\mathrm{m}})\} \quad \text{and} \quad \Psi_{\infty,0} := \Psi_\infty \cap \mathrm{H}^1_0(\Omega),$$

and notice that if $\psi \in \Psi_{\infty,0}$, then $\psi|_{\Omega_{\mathrm{f}}} \in \mathrm{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$ and $\psi|_{\Omega_{\mathrm{m}}} \in \mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}}) \cap \mathrm{L}^\infty(\Omega_{\mathrm{m}})$. Then, we let $\psi \in \Psi_{\infty,0}$, and multiply (2.1d) and (2.2c) by $\psi|_{\Omega_{\mathrm{f}}}$ and $\psi|_{\Omega_{\mathrm{m}}}$, respectively, to obtain

$$\kappa_{\mathrm{f}}\,(\nabla\theta_{\mathrm{f}}, \nabla\psi)_{\Omega_{\mathrm{f}}} - \kappa_{\mathrm{f}}\,\langle \nabla\theta_{\mathrm{f}} \cdot \mathbf{n}, \psi \rangle_\Sigma + (\mathbf{u}_{\mathrm{f}} \cdot \nabla\theta_{\mathrm{f}}, \psi)_{\Omega_{\mathrm{f}}} = 0,$$

and

$$\kappa_{\mathrm{m}}\,(\nabla\theta_{\mathrm{m}}, \nabla\psi)_{\Omega_{\mathrm{m}}} + \kappa_{\mathrm{m}}\,\langle \nabla\theta_{\mathrm{m}} \cdot \mathbf{n}, \psi \rangle_\Sigma + (\mathbf{u}_{\mathrm{m}} \cdot \nabla\theta_{\mathrm{m}}, \psi)_{\Omega_{\mathrm{m}}} = 0,$$

and summing up both equations, and using the interface condition (2.4b), we finally get

$$\kappa_{\mathrm{f}}\,(\nabla\theta_{\mathrm{f}}, \nabla\psi)_{\Omega_{\mathrm{f}}} + \kappa_{\mathrm{m}}\,(\nabla\theta_{\mathrm{m}}, \nabla\psi)_{\Omega_{\mathrm{m}}} + (\mathbf{u}_{\mathrm{f}} \cdot \nabla\theta_{\mathrm{f}}, \psi)_{\Omega_{\mathrm{f}}} + (\mathbf{u}_{\mathrm{m}} \cdot \nabla\theta_{\mathrm{m}}, \psi)_{\Omega_{\mathrm{m}}} = 0, \quad (3.1c)$$

for all $\psi \in \Psi_\infty$.

Finally, we incorporate the equations (2.1c), (2.2b), and (2.4c), weakly as follows

$$(q_{\mathrm{f}}, \mathrm{div}\,\mathbf{u}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} = 0, \quad (q_{\mathrm{m}}, \mathrm{div}\,\mathbf{u}_{\mathrm{m}})_{\Omega_{\mathrm{m}}} = 0 \quad \text{and} \quad \langle \mathbf{u}_{\mathrm{f}} \cdot \mathbf{n} - \mathbf{u}_{\mathrm{m}} \cdot \mathbf{n}, \xi \rangle_\Sigma = 0, \quad (3.1d)$$

for all $q_{\mathrm{f}} \in \mathrm{L}^2(\Omega_{\mathrm{f}})$, $q_{\mathrm{m}} \in \mathrm{L}^2(\Omega_{\mathrm{m}})$, and $\xi \in \mathrm{H}^{1/2}(\Sigma)$, respectively.

As a consequence of the above, we define $p := p_{\mathrm{f}}\chi_{\mathrm{f}} + p_{\mathrm{m}}\chi_{\mathrm{m}}$, $\theta := \theta_{\mathrm{f}}\chi_{\mathrm{f}} + \theta_{\mathrm{m}}\chi_{\mathrm{m}}$, with $\chi_\star$ being the characteristic function:

$$\chi_\star := \begin{cases} 1 & \text{in} \quad \Omega_\star, \\ 0 & \text{in} \quad \Omega \setminus \overline{\Omega}_\star, \end{cases}$$

for $\star \in \{\mathrm{f}, \mathrm{m}\}$, to obtain the variational problem: Find $\mathbf{u}_{\mathrm{f}} \in \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$, $\mathbf{u}_{\mathrm{m}} \in \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}})$, $p \in \mathrm{L}^2(\Omega)$, $\lambda \in \mathrm{H}^{1/2}(\Sigma)$ and $\theta \in \mathrm{H}^1(\Omega)$, with $\theta|_\Gamma = \theta_{\mathrm{D}}$, such that (3.1a)–(3.1d) hold.

Note that in principle the space for p_{m} does not allow enough regularity for the trace $p_{\mathrm{m}}|_{\Sigma}$ to exist. However, solution of (2.2a) has this unknown in $\mathrm{H}^1(\Omega_{\mathrm{m}})$ which justify the introduction of $\lambda \in \mathrm{H}^{1/2}(\Sigma)$. On the other hand, we observe that $\theta \in \mathrm{H}^1(\Omega)$ if and only if (2.4a) holds, so the interface condition (2.4a) is imposed on the temperature space. In turn, we notice that since $\mathbf{u}_{\mathrm{m}} \in \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}})$ and $\nabla\theta|_{\Omega_{\mathrm{m}}} = \nabla\theta_{\mathrm{m}} \in \mathbf{L}^2(\Omega_{\mathrm{m}})$, then $\mathbf{u}_{\mathrm{m}} \cdot \nabla\theta_{\mathrm{m}} \in \mathrm{L}^1(\Omega)$, which justify the introduction of the space $\Psi_{\infty,0}$ for the test function ψ in (3.1c).

Now, let us observe that if $(\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}}, p, \lambda, \theta)$ is a solution of the resulting variational problem, then for all $c \in \mathbb{R}$, $(\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}}, p + c, \lambda + c, \theta)$ is also a solution. Consequently, we avoid the non-uniqueness of (3.1a)–(3.1d) by requiring from now on that $p \in \mathrm{L}^2_0(\Omega)$.

In this way, we let:

$$\mathbf{u} := (\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}}) \in \mathbf{H} := \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}}),$$
$$(p, \lambda) \in \mathrm{Q} := \mathrm{L}^2_0(\Omega) \times \mathrm{H}^{1/2}(\Sigma),$$

where $\mathbf{H}$ and Q are endowed with the norms

$$\|\mathbf{v}\|_{\mathbf{H}} := \|\mathbf{v}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} + \|\mathbf{v}_{\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} \qquad \forall\, \mathbf{v} \in \mathbf{H},$$
$$\|(q,\xi)\|_{\mathrm{Q}} := \|q\|_{0,\Omega} + \|\xi\|_{1/2,\Sigma} \qquad \forall\, (q,\xi) \in \mathrm{Q},$$

and arrive at the following variational problem:

Find $(\mathbf{u}, (p,\lambda)) \in \mathbf{H} \times \mathrm{Q}$ and $\theta \in \mathrm{H}^1(\Omega)$, with $\theta|_{\Gamma} = \theta_{\mathrm{D}}$, such that

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) + B(\mathbf{v},(p,\lambda)) - D(\theta,\mathbf{v}) &= 0, \\ B(\mathbf{u},(q,\xi)) &= 0, \\ A_{\mathrm{T}}(\theta,\psi) + O_{\mathrm{T}}(\mathbf{u};\theta,\psi) &= 0, \end{aligned} \tag{3.2}$$

for all $\mathbf{v} = (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{H}$, for all $(q,\xi) \in \mathrm{Q}$ and for all $\psi \in \Psi_{\infty,0}$, where the forms $A_{\mathrm{F}} : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$, $O_{\mathrm{F}} : \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \to \mathbb{R}$, $B : \mathbf{H} \times \mathrm{Q} \to \mathbb{R}$, $D : \mathrm{H}^1(\Omega) \times \mathbf{H} \to \mathbb{R}$, $A_{\mathrm{T}} : \mathrm{H}^1(\Omega) \times \Psi_{\infty,0} \to \mathbb{R}$, and $O_{\mathrm{T}} : \mathbf{H} \times \mathrm{H}^1(\Omega) \times \Psi_{\infty,0} \to \mathbb{R}$, are defined respectively, as

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) &:= a_{\mathrm{F,f}}(\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) + a_{\mathrm{F,m}}(\mathbf{u}_{\mathrm{m}},\mathbf{v}_{\mathrm{m}}), \\ O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) &:= ((\mathbf{w}_{\mathrm{f}} \cdot \nabla)\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}}, \\ B(\mathbf{v},(q,\xi)) &:= -(q,\mathrm{div}\,\mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} - (q,\mathrm{div}\,\mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}} + \langle \mathbf{v}_{\mathrm{f}} \cdot \mathbf{n} - \mathbf{v}_{\mathrm{m}} \cdot \mathbf{n}, \xi\rangle_{\Sigma}, \\ D(\theta,\mathbf{v}) &:= (\theta\,\mathbf{g}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} + (\theta\,\mathbf{g}_{\mathrm{m}},\mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}}, \\ A_{\mathrm{T}}(\theta,\psi) &:= \kappa_{\mathrm{f}}(\nabla\theta,\nabla\psi)_{\Omega_{\mathrm{f}}} + \kappa_{\mathrm{m}}(\nabla\theta,\nabla\psi)_{\Omega_{\mathrm{m}}}, \\ O_{\mathrm{T}}(\mathbf{w};\theta,\psi) &:= (\mathbf{w}_{\mathrm{f}} \cdot \nabla\theta,\psi)_{\Omega_{\mathrm{f}}} + (\mathbf{w}_{\mathrm{m}} \cdot \nabla\theta,\psi)_{\Omega_{\mathrm{m}}}, \end{aligned} \tag{3.3}$$

with

$$
\begin{aligned}
a_{\mathrm{F,f}}(\mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f}) \quad &:= \; 2\mu \, (\mathbf{e}(\mathbf{u}_\mathrm{f}), \mathbf{e}(\mathbf{v}_\mathrm{f}))_{\Omega_\mathrm{f}} \; + \sum_{j=1}^{d-1} \omega_j \, \langle \mathbf{u}_\mathrm{f} \cdot \mathbf{t}_j, \mathbf{v}_\mathrm{f} \cdot \mathbf{t}_j \rangle_{\Sigma} \, , \\
a_{\mathrm{F,m}}(\mathbf{u}_\mathrm{m}, \mathbf{v}_\mathrm{m}) \quad &:= \; (\mathbf{K}^{-1} \, \mathbf{u}_\mathrm{m}, \mathbf{v}_\mathrm{m})_{\Omega_\mathrm{m}} .
\end{aligned}
$$

3.2.2 Existence and stability of solution

Now we address the existence and stability of solution of problem (3.2). We start the analysis by deriving the stability properties of the forms involved. More precisely, in what follows we derive the continuity of each form involved in the problem, an inf-sup estimate for the bilinear form B and the ellipticity of the bilinear form $A_\mathrm{F}(\cdot,\cdot) + O_\mathrm{F}(\mathbf{w}_\mathrm{f};\cdot,\cdot)$, for a given $\mathbf{w}_\mathrm{f} \in \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$. This estimates will be required to prove existence and uniqueness of solution of problem (3.2).

Stability properties

We begin by observing that, after simple computations, the bilinear forms A_F, B and A_T are bounded, that is,

$$
\begin{aligned}
|A_\mathrm{F}(\mathbf{u}, \mathbf{v})| \; &\leq \; C_{A_\mathrm{F}} \, \|\mathbf{u}\|_{\mathbf{H}} \, \|\mathbf{v}\|_{\mathbf{H}} && \forall \, \mathbf{u}, \mathbf{v} \in \mathbf{H}, && (3.4) \\
|B(\mathbf{v}, (q,\xi))| \; &\leq \; C_B \|\mathbf{v}\|_{\mathbf{H}} \, \|(q,\xi)\|_{\mathrm{Q}} && \forall \, \mathbf{v} \in \mathbf{H}, \forall \, (q,\xi) \in \mathrm{Q}, && \\
|A_\mathrm{T}(\theta, \psi)| \; &\leq \; C_{A_\mathrm{T}} \, \|\theta\|_{1,\Omega} \, \|\psi\|_{1,\Omega} && \forall \, \theta \in \mathrm{H}^1(\Omega), \forall \, \psi \in \Psi_\infty . && (3.5)
\end{aligned}
$$

In turn, employing the Hölder and Sobolev inequalities (cf. (1.1) and (1.7), respectively), it is easy to see that

$$
|D(\theta, \mathbf{v})| \; \leq \; C_D \left(\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}} \right) \|\theta\|_{1,\Omega} \, \|\mathbf{v}\|_{\mathbf{H}} \qquad \forall \, \theta \in \Psi_\infty, \, \forall \, \mathbf{v} \in \mathbf{H}. \tag{3.6}
$$

Similarly, for O_F we utilize again (1.1) and (1.7), to obtain

$$
|O_\mathrm{F}(\mathbf{w}_\mathrm{f}; \mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f})| \; \leq \; C_{O_\mathrm{F}} \, \|\mathbf{w}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \, \|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \, \|\mathbf{v}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \quad \forall \, \mathbf{w}_\mathrm{f}, \mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f} \in \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}).
$$

Now, we let $\mathbf{V}$ be the kernel of the bilinear form B, that is

$$
\mathbf{V} := \{ \mathbf{v} \in \mathbf{H} : B(\mathbf{v}, (q,\xi)) = 0 \quad \forall \, (q,\xi) \in \mathrm{Q} \}. \tag{3.7}
$$

From the definition of B we observe that $\mathbf{v} = (\mathbf{v}_\mathrm{f}, \mathbf{v}_\mathrm{m}) \in \mathbf{V}$ if and only if

$$
(q, \operatorname{div} \mathbf{v}_\mathrm{f})_{\Omega_\mathrm{f}} + (q, \operatorname{div} \mathbf{v}_\mathrm{m})_{\Omega_\mathrm{m}} = 0 \qquad \forall \, q \in \mathrm{L}^2_0(\Omega),
$$

and

$$
\langle \mathbf{v}_\mathrm{f} \cdot \mathbf{n} - \mathbf{v}_\mathrm{m} \cdot \mathbf{n}, \xi \rangle_{\Sigma} = 0 \qquad \forall \, \xi \in \mathrm{H}^{1/2}(\Sigma).
$$

Then, noting that $\mathrm{L}^2(\Omega) = \mathrm{L}_0^2(\Omega) \oplus \mathbb{R}$, and taking $\xi \in \mathbb{R}$ in the latter equation, we deduce that

$$(q, \operatorname{div} \mathbf{v}_\mathrm{f})_{\Omega_\mathrm{f}} + (q, \operatorname{div} \mathbf{v}_\mathrm{m})_{\Omega_\mathrm{m}} = 0 \qquad \forall\, q \in \mathrm{L}^2(\Omega),$$

which implies

$$\operatorname{div} \mathbf{v}_\mathrm{f} = 0 \quad \text{in} \quad \Omega_\mathrm{f} \quad \text{and} \quad \operatorname{div} \mathbf{v}_\mathrm{m} = 0 \quad \text{in} \quad \Omega_\mathrm{m}.$$

According to the above, we can rewrite $\mathbf{V}$ as

$$\mathbf{V} := \{\mathbf{v} = (\mathbf{v}_\mathrm{f}, \mathbf{v}_\mathrm{m}) \in \mathbf{V}_\mathrm{f} \times \mathbf{V}_\mathrm{m} : \mathbf{v}_\mathrm{f} \cdot \mathbf{n} - \mathbf{v}_\mathrm{m} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Sigma\},$$

with

$$\begin{aligned}
\mathbf{V}_\mathrm{f} &:= \left\{\mathbf{v}_\mathrm{f} \in \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}) : \operatorname{div} \mathbf{v}_\mathrm{f} = 0 \quad \text{in} \quad \Omega_\mathrm{f}\right\}, \\
\mathbf{V}_\mathrm{m} &:= \{\mathbf{v}_\mathrm{m} \in \mathbf{H}_{\Gamma_\mathrm{m}}(\operatorname{div}; \Omega_\mathrm{m}) : \operatorname{div} \mathbf{v}_\mathrm{m} = 0 \quad \text{in} \quad \Omega_\mathrm{m}\}.
\end{aligned}$$

Next, we employ the well-known Korn's inequality for the bilinear form $a_{\mathrm{F,f}}$ (cf. (1.4)), and the fact that $\mathbf{K}^{-1}$ is symmetric and positive definite (cf. (2.3)) for $a_{\mathrm{F,m}}$, to deduce that

$$a_{\mathrm{F,f}}(\mathbf{v}_\mathrm{f}, \mathbf{v}_\mathrm{f}) \,\geq\, 2\mu\alpha_\mathrm{f}\, \|\mathbf{v}_\mathrm{f}\|^2_{1,\Omega_\mathrm{f}} \quad \text{and} \quad a_{\mathrm{F,m}}(\mathbf{v}_\mathrm{m}, \mathbf{v}_\mathrm{m}) \,\geq\, C_\mathbf{K}\, \|\mathbf{v}_\mathrm{m}\|^2_{\operatorname{div};\Omega_\mathrm{m}}, \tag{3.8}$$

for all $\mathbf{v}_\mathrm{f} \in \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, and for all $\mathbf{v}_\mathrm{m} \in \mathbf{V}_\mathrm{m}$, with $\alpha_\mathrm{f} > 0$.

Using these estimates we deduce that the form $A_\mathrm{F}(\cdot,\cdot) + O_\mathrm{F}(\mathbf{w}_\mathrm{f};\cdot,\cdot) : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$, is elliptic on $\mathbf{V}$ for suitable $\mathbf{w}_\mathrm{f} \in \mathbf{V}_\mathrm{f}$. More precisely, we have the following lemma. For its proof, we refer the reader to [45, Lemma 2].

Lemma 3.2.1. *Let $\mathbf{w}_\mathrm{f} \in \mathbf{V}_\mathrm{f}$ be such that*

$$\|\mathbf{w}_\mathrm{f} \cdot \mathbf{n}\|_{0,\Sigma} \,\leq\, \frac{2\mu\alpha_\mathrm{f}}{C^2_{\mathrm{tr}} C^2_{\mathrm{Sob},\Sigma}}, \tag{3.9}$$

where $C_{\mathrm{tr}} > 0$ is the constant of the well-known trace inequality (cf. Theorem 1.6.4). There holds

$$A_\mathrm{F}(\mathbf{v}, \mathbf{v}) \,+\, O_\mathrm{F}(\mathbf{w}_\mathrm{f}; \mathbf{v}_\mathrm{f}, \mathbf{v}_\mathrm{f}) \,\geq\, \alpha_\mathrm{F}\, \|\mathbf{v}\|^2_\mathbf{H} \qquad \forall\, \mathbf{v} \in \mathbf{V}, \tag{3.10}$$

with $\alpha_\mathrm{F} := \frac{1}{2} \min\{\mu\alpha_\mathrm{f}, C_\mathbf{K}\}$.

We also recall from [45, Lemma 1] that the bilinear form B satisfies the following inf-sup condition

$$\sup_{\mathbf{v} \in \mathbf{H}\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v}, (q, \xi))}{\|\mathbf{v}\|_\mathbf{H}} \,\geq\, \beta\, \|(q, \xi)\|_\mathrm{Q} \qquad \forall\, (q, \xi) \in \mathrm{Q}, \tag{3.11}$$

with $\beta > 0$.

We continue by introducing the following lemma that summarizes some properties of the form O_T.

Lemma 3.2.2. *The following identity holds true:*

$$O_{\mathrm{T}}(\mathbf{w};\theta,\psi) \;=\; -\,O_{\mathrm{T}}(\mathbf{w};\psi,\theta) \qquad \forall\,\mathbf{w}\in\mathbf{V},\ \ \forall\,\theta,\psi\in\Psi_\infty. \tag{3.12}$$

In addition, for all $\mathbf{w}\in\mathbf{H}$ *and for all* $\theta,\psi\in\Psi_\infty$*, there exist positive constants* $C_{O_{\mathrm{T}}},\widetilde{C}_{O_{\mathrm{T}}}$*, such that*

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \;\le\; C_{O_{\mathrm{T}}}\|\mathbf{w}\|_{\mathbf{H}}\|\psi\|_{1,\Omega}\left(\|\theta\|_{0,3,\Omega_{\mathrm{f}}}+\|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}\right), \tag{3.13a}$$

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \;\le\; \widetilde{C}_{O_{\mathrm{T}}}\,\|\mathbf{w}\|_{\mathbf{H}}\,\|\psi\|_{1,\Omega}\,\left(\|\theta\|_{1,\Omega_{\mathrm{f}}}\,+\,\|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}\right). \tag{3.13b}$$

Finally, there exists $\overline{C}_{O_{\mathrm{T}}}>0$*, such that*

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \;\le\; \overline{C}_{O_{\mathrm{T}}}\|\mathbf{w}\|_{\mathbf{H}}\|\psi\|_{1,\Omega}\left(\|\theta\|_{1,\Omega_{\mathrm{f}}}+\|\theta\|_{1,4,\Omega_{\mathrm{m}}}\right), \tag{3.14}$$

for all $\mathbf{w}\in\mathbf{H}$*, for all* $\psi\in\mathrm{H}^1(\Omega)$ *and for all* $\theta\in\mathrm{H}^1(\Omega)$ *such that* $\theta|_{\Omega_{\mathrm{m}}}\in\mathrm{W}^{1,4}(\Omega_{\mathrm{m}})$.

Proof. Let $\mathbf{w}\in\mathbf{V}$ and $\theta,\psi\in\Psi_{\infty,0}$ be given. Noticing that $\psi|_{\Omega_{\mathrm{m}}}\in\mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})\cap\mathrm{L}^\infty(\Omega_{\mathrm{m}})$, it readily follows that

$$\nabla(\theta|_{\Omega_{\mathrm{m}}}\psi|_{\Omega_{\mathrm{m}}})=\theta|_{\Omega_{\mathrm{m}}}\nabla(\psi|_{\Omega_{\mathrm{m}}})+\psi|_{\Omega_{\mathrm{m}}}\nabla(\theta|_{\Omega_{\mathrm{m}}})\in\mathbf{L}^2(\Omega_{\mathrm{m}}).$$

Then, integrating by parts the two terms defining the form O_{T} (cf. (3.3)) and using the fact that $\operatorname{div}\mathbf{w}_\star=0$ in $\Omega_\star$, $\mathbf{w}_\star\cdot\mathbf{n}=0$ on $\Gamma_\star$ for $\star\in\{\mathrm{f},\mathrm{m}\}$, and $\mathbf{w}_{\mathrm{f}}\cdot\mathbf{n}-\mathbf{w}_{\mathrm{m}}\cdot\mathbf{n}=0$ on Σ, we easily obtain (3.12).

Now, for (3.13a) we employ (3.12) and (1.1), to obtain

$$\begin{aligned}
|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \;&=\; |O_{\mathrm{T}}(\mathbf{w};\psi,\theta)|\\
&\le |(\mathbf{w}_{\mathrm{f}}\cdot\nabla\psi,\theta)_{\Omega_{\mathrm{f}}}|\,+\,|(\mathbf{w}_{\mathrm{m}}\cdot\nabla\psi,\theta)_{\Omega_{\mathrm{m}}}|\\
&\le \|\mathbf{w}_{\mathrm{f}}\|_{0,6,\Omega_{\mathrm{f}}}|\psi|_{1,\Omega_{\mathrm{f}}}\,\|\theta\|_{0,3,\Omega_{\mathrm{f}}}+\|\mathbf{w}_{\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}}|\psi|_{1,\Omega_{\mathrm{m}}}\|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}.
\end{aligned} \tag{3.15}$$

Then, applying (1.7) to $\|\mathbf{w}_{\mathrm{f}}\|_{0,6,\Omega_{\mathrm{f}}}$ we easily deduce (3.13a). In turn, applying (1.7) to $\|\theta\|_{0,3,\Omega_{\mathrm{f}}}$, estimate (3.13b) follows easily from (3.13a).

Finally, to deduce (3.14), we apply (1.1), to obtain

$$\begin{aligned}
|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \;&\le |(\mathbf{w}_{\mathrm{f}}\cdot\nabla\theta,\psi)_{\Omega_{\mathrm{f}}}|\,+\,|(\mathbf{w}_{\mathrm{m}}\cdot\nabla\theta,\psi)_{\Omega_{\mathrm{m}}}|\\
&\le \|\mathbf{w}_{\mathrm{f}}\|_{0,4,\Omega_{\mathrm{f}}}\,|\theta|_{1,\Omega_{\mathrm{f}}}\,\|\psi\|_{0,4,\Omega_{\mathrm{f}}}\,+\,\|\mathbf{w}_{\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}}\,|\theta|_{1,4,\Omega_{\mathrm{m}}}\,\|\psi\|_{0,4,\Omega_{\mathrm{m}}}.
\end{aligned}$$

In this way, applying the Sobolev inequality (1.7) to $\|\mathbf{w}_{\mathrm{f}}\|_{0,4,\Omega_{\mathrm{f}}}$, $\|\psi\|_{0,4,\Omega_{\mathrm{f}}}$ and $\|\psi\|_{0,4,\Omega_{\mathrm{m}}}$, we get (3.14), which concludes the proof. $\square$

In (3.15), it's noteworthy that $\mathbf{w}_{\rm f}$ and θ could have been estimated using the norms $\|\cdot\|_{0,p,\Omega_{\rm f}}$ and $\|\cdot\|_{0,q,\Omega_{\rm f}}$, respectively, for any $p \in [1,6]$ and $q \geq 1$ satisfying $1/p + 1/q = 1/2$. However, we opt to set $q = 3$ for consistency, as it aligns with our subsequent proof of Lemma 3.2.6. There, we combine (3.13a) and (3.23a) to deduce that, for a sufficiently small parameter $\delta > 0$ (to be specified later in Section 3.2.2), estimate (3.40) hold.

On the other hand, using the Poincaré inequality (cf. (1.3)), we obtain

$$A_{\rm T}(\psi,\psi) \,\geq\, \alpha_{\rm T}\,\|\psi\|_{1,\Omega}^2 \qquad \forall\,\psi \in {\rm H}_0^1(\Omega), \tag{3.16}$$

with constant $\alpha_{\rm T} := C_{\rm P}\min\{\kappa_{\rm f}\,,\kappa_{\rm m}\}$, where $C_{\rm P} > 0$ is the corresponding Poincaré's constant. Combining this estimate and (3.12) one easily deduce that for a given $\mathbf{w} \in \mathbf{V}$, $A_{\rm T}(\cdot,\cdot) + O_{\rm T}(\mathbf{w};\cdot,\cdot)$ is elliptic on $\Psi_{\infty,0}$, that is

$$A_{\rm T}(\psi,\psi) \,+\, O_{\rm T}(\mathbf{w};\psi,\psi) \,\geq\, \alpha_{\rm T}\,\|\psi\|_{1,\Omega}^2 \qquad \forall\,\psi \in \Psi_{\infty,0}. \tag{3.17}$$

An equivalent reduced problem

To simplify the analysis of existence and stability of solutions of (3.2), we now introduce a reduced equivalent version of problem. More precisely, in what follows we restrict problem (3.2) to the kernel of the bilinear form B, namely $\mathbf{V}$, and decompose the temperature by means of a suitable lifting operator. Then, we obtain an equivalent reduced problem that can be analyzed by means of classical results, such as the Brouwer's fixed point theorem and the Lax–Milgram lemma. We begin by introducing the aforementioned lifting operator. To do this, we let ${\rm E} : {\rm W}^{3/4,4}(\Gamma) \to {\rm W}^{1,4}(\Omega)$ be the usual lifting operator (see for instance [47, Corollary B.53]), satisfying

$$\gamma_0({\rm E}(\zeta)) = \zeta \quad \text{and} \quad \|{\rm E}(\zeta)\|_{1,4,\Omega} \leq c\|\zeta\|_{3/4,4,\Gamma} \quad \forall\,\zeta \in {\rm W}^{3/4,4}(\Gamma), \tag{3.18}$$

where $\gamma_0 : {\rm W}^{1,4}(\Omega) \to {\rm W}^{3/4,4}(\Gamma)$ is the trace operator. In turn, we let $\delta > 0$ and, similarly to [9, Lemma 2.8], define the function $\beta_\delta : \mathbb{R}^d \to \mathbb{R}$ given by

$$\beta_\delta(\mathbf{x}) := \begin{cases} 1 & \text{if } 0 \leq \text{dist}(\mathbf{x},\Gamma) \leq \delta, \\ 2 - \delta^{-1}\text{dist}(\mathbf{x},\Gamma) & \text{if } \delta \leq \text{dist}(\mathbf{x},\Gamma) \leq 2\delta, \\ 0 & \text{if } \text{dist}(\mathbf{x},\Gamma) \geq 2\delta, \end{cases} \tag{3.19}$$

where $\text{dist}(\mathbf{x},\Gamma)$ denotes the distance from the point $\mathbf{x}$ to the boundary Γ. Observe that β_δ is continuous and satisfies the following properties

$$\begin{aligned} & \beta_\delta \in {\rm W}^{1,\infty}(\Omega), \\ & 0 \leq \beta_\delta \leq 1 \quad \text{in} \quad \Omega_\delta, \\ & \beta_\delta \equiv 0 \quad \text{in} \quad \Omega\setminus\Omega_\delta, \\ & \|\nabla\beta_\delta\|_{0,4,\Omega_\delta} \leq \delta^{-1}|\Omega_\delta|^{1/4}, \end{aligned} \tag{3.20}$$

where $\Omega_\delta := \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \Gamma) < 2\delta\}$, which satisfies $|\Omega_\delta| \le C_\Gamma \delta$, with C_Γ being a positive constant that depends on the measure of Γ. In this way, in order to handle the non-homogeneous Dirichlet boundary condition for the temperature, we introduce the extension operator

$$\mathrm{E}_\delta := \beta_\delta \mathrm{E} : \mathrm{W}^{3/4,4}(\Gamma) \to \mathrm{W}^{1,4}(\Omega). \tag{3.21}$$

In the following lemma, we summarize some properties of this operator.

Lemma 3.2.3. *For all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$, $\mathrm{E}_\delta(\zeta)$ lies in $\mathrm{L}^\infty(\Omega)$ and satisfies the estimate*

$$\|\mathrm{E}_\delta(\zeta)\|_{0,\infty,\Omega} \le \|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta}. \tag{3.22}$$

In addition, there exist $C_{\mathrm{lift},1}$, $C_{\mathrm{lift},2}$, $\mathrm{C}_{\mathrm{lift},3} > 0$, such that

$$\|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} \le C_{\mathrm{lift},1}\, \delta^{1/12}\, \|\zeta\|_{3/4,4,\Gamma} \qquad \forall \zeta \in \mathrm{W}^{3/4,4}(\Gamma), \tag{3.23a}$$

$$\|\mathrm{E}_\delta(\zeta)\|_{1,\Omega} \le C_{\mathrm{lift},2}\, \delta^{-3/4}(1+\delta^2)^{1/2}\, \|\zeta\|_{3/4,4,\Gamma} \quad \forall \zeta \in \mathrm{W}^{3/4,4}(\Gamma), \tag{3.23b}$$

$$\|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega} \le C_{\mathrm{lift},3}\, (1+\delta^{-2})^{1/2} \|\zeta\|_{3/4,4,\Gamma} \qquad \forall \zeta \in \mathrm{W}^{3/4,4}(\Gamma). \tag{3.23c}$$

Proof. We begin the proof by observing that the first Sobolev's embedding in (1.6) with $q = 1$ and $r = 4$ guarantees the fact that $\mathrm{E}_\delta(\zeta) \in \mathrm{C}^0(\overline{\Omega}) \subseteq \mathrm{L}^\infty(\Omega)$, for all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$. In addition, using the second and third properties of β_δ (cf. (3.19)) given in (3.20), there holds

$$\|\mathrm{E}_\delta(\zeta)\|_{0,\infty,\Omega} = \|\mathrm{E}_\delta(\zeta)\|_{0,\infty,\Omega_\delta} \le \|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta},$$

which implies (3.22).

Now, to derive (3.23a) and (3.23b) we proceed similarly to [37, Lemma 3.2]. In fact, we first apply the Hölder inequality (1.1) with $p = 4$ and $q = 4/3$ and the estimate in (3.18), to obtain

$$\begin{aligned} \|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega}^3 &\le \|\mathrm{E}(\zeta)\|_{0,3,\Omega_\delta}^3 \le |\Omega_\delta|^{1/4} \|\mathrm{E}(\zeta)\|_{0,4,\Omega}^3 \\ &\le C_\Gamma^{1/4} \delta^{1/4}\, \|\mathrm{E}(\zeta)\|_{1,4,\Omega}^3 \le c\, \delta^{1/4} \|\zeta\|_{3/4,4,\Gamma}^3, \end{aligned}$$

which implies (3.23a). Similarly, but now applying Hölder's inequality with $p = q = 2$ and the properties of β_δ in (3.20), it is follows that

$$\begin{aligned} \|\nabla(\mathrm{E}_\delta(\zeta))\|_{0,\Omega} &\le \|\nabla(\beta_\delta)\mathrm{E}(\zeta)\|_{0,\Omega_\delta} + \|\beta_\delta \nabla(\mathrm{E}(\zeta))\|_{0,\Omega_\delta} \\ &\le c\, \delta^{-1}\, |\Omega_\delta|^{1/4}\, \|\mathrm{E}(\zeta)\|_{0,4,\Omega} + |\Omega_\delta|^{1/4}\, \|\nabla(\mathrm{E}(\zeta))\|_{0,4,\Omega} \\ &\le c\, \delta^{-3/4}(1+\delta^2)^{1/2}\, \|\zeta\|_{3/4,4,\Gamma}, \end{aligned} \tag{3.24}$$

which gives (3.23b).

Finally, to deduce (3.23c) we first observe that estimate (3.18) and the properties of β_δ in (3.20), imply

$$\|\mathrm{E}_\delta(\zeta)\|_{0,4,\Omega} = \|\beta_\delta \mathrm{E}(\zeta)\|_{0,4,\Omega} \leq \|\mathrm{E}(\zeta)\|_{0,4,\Omega} \leq c\|\zeta\|_{3/4,4,\Gamma}.$$

In turn, using the fact that $|\nabla\beta_\delta| = \delta^{-1}$ a.e in $\{\mathbf{x} \in \Omega_\delta : \delta \leq \mathrm{dist}(\mathbf{x},\Gamma) \leq 2\delta\}$ and $\nabla\beta_\delta$ vanishes elsewhere, and proceeding similarly to (3.24), we obtain

$$\begin{aligned}\|\nabla(\mathrm{E}_\delta(\zeta))\|_{0,4,\Omega} &\leq \|\nabla(\beta_\delta)\mathrm{E}(\zeta)\|_{0,4,\Omega_\delta} + \|\beta_\delta\nabla(\mathrm{E}(\zeta))\|_{0,4,\Omega_\delta}\\ &\leq c\,\delta^{-1}\,\|\mathrm{E}(\zeta)\|_{0,4,\Omega} + \|\nabla(\mathrm{E}(\zeta))\|_{0,4,\Omega}\\ &\leq c\,(1+\delta^{-2})^{1/2}\,\|\zeta\|_{3/4,4,\Gamma}.\end{aligned}$$

Using the estimates above we readily deduce (3.23c), which concludes the proof. □

Given a fixed $\delta > 0$ now we define the following lifting for the Dirichlet datum $\theta_\mathrm{D} \in \mathrm{W}^{3/4,4}(\Gamma)$:

$$\theta_1 \,:=\, \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega), \tag{3.25}$$

and decompose the unknown $\theta \in \mathrm{H}^1(\Omega)$ as $\theta = \theta_0 + \theta_1$, with $\theta_0 \in \mathrm{H}^1_0(\Omega)$. In turn, we recall from the second equation of (3.2) that the unknown $\mathbf{u} = (\mathbf{u}_\mathrm{f},\mathbf{u}_\mathrm{m}) \in \mathbf{H}$ satisfies $B(\mathbf{u},(q,\xi)) = 0 \quad \forall\,(q,\xi) \in \mathrm{Q}$, which implies that $\mathbf{u} \in \mathbf{V}$ (cf. (3.7)).

According to the above, now we introduce the reduced version of problem (3.2) on the kernel $\mathbf{V}$, which consists in finding $(\mathbf{u},\theta_0) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$, such that

$$\begin{aligned} A_\mathrm{F}(\mathbf{u},\mathbf{v}) \,+\, O_\mathrm{F}(\mathbf{u}_\mathrm{f};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) \,-\, D(\theta_0,\mathbf{v}) \;&=\; D(\theta_1,\mathbf{v}) \qquad && \forall\,\mathbf{v} \in \mathbf{V},\\ A_\mathrm{T}(\theta_0,\psi) \,+\, O_\mathrm{T}(\mathbf{u};\theta_0+\theta_1,\psi) \;&=\; -\,A_\mathrm{T}(\theta_1,\psi) \qquad && \forall\,\psi \in \Psi_{\infty,0}. \end{aligned} \tag{3.26}$$

It is not difficult to see that problems (3.26) and (3.2) are equivalent. This result is established next.

Lemma 3.2.4. *If* $(\mathbf{u},(p,\lambda),\theta) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{H}^1(\Omega)$ *is a solution of* (3.2)*, then* $\mathbf{u} \in \mathbf{V}$ *and* $(\mathbf{u},\theta_0) = (\mathbf{u},\theta-\theta_1)$*, with* θ_1 *defined in* (3.25) *is a solution to* (3.26)*. Conversely, if* $(\mathbf{u},\theta_0) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ *is a solution to* (3.26)*, then there exists* $p \in \mathrm{L}^2_0(\Omega)$ *and* $\lambda \in \mathrm{H}^{1/2}(\Sigma)$ *such that* $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1)$ *is a solution to* (3.2)*.*

Proof. The proof follows from the definition of the lifting θ_1 (cf. (3.25)) and the inf-sup condition (3.11). We omit further details and refer the reader to [85, Lemma 2.1] for a similar result. □

According to the previous lemma, to prove existence of solution of problem (3.2), it suffices to prove existence of solution of problem (3.26). In addition, by deriving the stability of solution of problem (3.26) one can easily obtain the corresponding stability for (3.2). In the following section we shall address these aspects.

Existence of solution and stability

In what follows we proceed similarly to [7, Section 2.3] to prove existence of solution of (3.26) by means of a Galerkin's method and a fixed-point strategy. More precisely, since the trial and test spaces are different, we introduce a Galerkin scheme for (3.26) to obtain a finite-dimensional square system of nonlinear equations. Then, we apply the Brouwer Fixed Point Theorem to prove existence of solution of the resulting finite-dimensional problem and pass to the limit to obtain the desired solution.

We begin by recalling from [16, Propositions 9.1 and 3.25] that $\mathrm{W}_0^{1,4}(\Omega)$ is separable and has a countable basis. In turn, since $\mathbf{V}$ is a closed subspace of $\mathbf{H} = \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}) \times \mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m})$ and $\mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$ and $\mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m})$ are separable, $\mathbf{V}$ is also separable and has a countable basis. Then, we let $\{\varphi_i\}_{i\in\mathbb{N}}$ and $\{\mathbf{z}_i\}_{i\in\mathbb{N}} := \{(\mathbf{z}_{i,\mathrm{f}},\mathbf{z}_{i,\mathrm{m}})\}_{i\in\mathbb{N}}$ be the countable bases of $\mathrm{W}_0^{1,4}(\Omega)$ and $\mathbf{V}$, respectively and for a fixed $n\in\mathbb{N}$, we let $\Psi_n = \langle\{\varphi_1,\dots,\varphi_n\}\rangle$ and $\mathbf{V}_n = \langle\{\mathbf{z}_1,\dots,\mathbf{z}_n\}\rangle$. We define the following finite-dimensional nonlinear problem:

Find $(\mathbf{u}_n,\theta_{n,0}) := ((\mathbf{u}_{n,\mathrm{f}},\mathbf{u}_{n,\mathrm{m}}),\theta_{n,0}) \in \mathbf{V}_n\times\Psi_n$, such that

$$
\begin{aligned}
A_\mathrm{F}(\mathbf{u}_n,\mathbf{v}) + O_\mathrm{F}(\mathbf{u}_{n,\mathrm{f}};\mathbf{u}_{n,\mathrm{f}},\mathbf{v}_\mathrm{f}) - D(\theta_{n,0},\mathbf{v}) &= D(\theta_1,\mathbf{v}),\\
A_\mathrm{T}(\theta_{n,0},\psi) + O_\mathrm{T}(\mathbf{u}_n;\theta_{n,0}+\theta_1,\psi) &= -A_\mathrm{T}(\theta_1,\psi),
\end{aligned}
\tag{3.27}
$$

for all $(\mathbf{v},\psi) := ((\mathbf{v}_\mathrm{f},\mathbf{v}_\mathrm{m}),\psi)\in\mathbf{V}_n\times\Psi_n$, with θ_1 defined as in (3.25).

Notice that (3.27) is a discrete version of (3.26) since $\mathrm{W}_0^{1,4}(\Omega)\subseteq\mathrm{L}^\infty(\Omega)$ and $\mathrm{W}_0^{1,4}(\Omega)\subseteq\mathrm{H}_0^1(\Omega)$ (owing to (1.6)).

In what follows, we prove that problem (3.27) has at least one solution by means of the classical Brouwer's fixed point theorem (cf. Theorem 1.6.9), which is rewritten below for convenience and readability.

Theorem 3.2.5. *Let* Y *be a compact and convex subset of a finite dimensional Banach space* X*, and let* $f:\mathrm{Y}\to\mathrm{Y}$ *be a continuous mapping. Then,* f *has at least one fixed point.*

To apply Theorem 3.2.5 to the context of problem (3.27), we first define the compact and convex set

$$
\mathbf{X}_n := \left\{ (\mathbf{w},\phi)\in\mathbf{V}_n\times\Psi_n : \quad \begin{array}{l} \|\mathbf{w}\|_\mathbf{H} \,\le\, C_\mathbf{u}\,(\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}})\|\theta_1\|_{1,\Omega} \\ \text{and } \|\phi\|_{1,\Omega} \,\le\, C_\theta\,\|\theta_1\|_{1,\Omega} \end{array} \right\},
$$

with $C_\mathbf{u}$ and C_θ given by

$$
C_\mathbf{u} := 2\,\alpha_\mathrm{F}^{-1}\,\alpha_\mathrm{T}^{-1}\,C_D\,\left(C_{A_\mathrm{T}} + \alpha_\mathrm{T}\right), \tag{3.28}
$$

and

$$
C_\theta := \alpha_\mathrm{T}^{-1}\left(2\,C_{A_\mathrm{T}} + \alpha_\mathrm{T}\right), \tag{3.29}
$$

respectively. In turn, we let $\mathcal{J}_n : \mathbf{X}_n \to \mathbf{V}_n \times \Psi_n$ be the operator defined by

$$\mathcal{J}_n(\mathbf{w},\phi) = (\mathbf{u}_n, \theta_{n,0}) \qquad \forall\, (\mathbf{w},\phi) = ((\mathbf{w}_\mathrm{f}, \mathbf{w}_\mathrm{m}), \phi) \in \mathbf{X}_n, \tag{3.30}$$

where $(\mathbf{u}_n, \theta_{n,0})$ is the unique solution (to be verified below) of the linearized version of problem (3.27): Find $(\mathbf{u}_n, \theta_{n,0}) \in \mathbf{V}_n \times \Psi_n$, such that

$$\begin{aligned} A_\mathrm{F}(\mathbf{u}_n, \mathbf{v}) \,+\, O_\mathrm{F}(\mathbf{w}_\mathrm{f}; \mathbf{u}_{n,\mathrm{f}}, \mathbf{v}_\mathrm{f}) \;&=\; D(\phi, \mathbf{v}) \,+\, D(\theta_1, \mathbf{v}) \\ A_\mathrm{T}(\theta_{n,0}, \psi) \,+\, O_\mathrm{T}(\mathbf{w}; \theta_{n,0}, \psi) \;&=\; -\,A_\mathrm{T}(\theta_1, \psi) \,-\, O_\mathrm{T}(\mathbf{w}; \theta_1, \psi), \end{aligned} \tag{3.31}$$

for all $\mathbf{v} \in \mathbf{V}_n$ and for all $\psi \in \Psi_n$. Then, it is clear that $(\mathbf{u}_n, \theta_{n,0}) \in \mathbf{V}_n \times \Psi_n$ is a solution of problem (3.27), if and only if, $\mathcal{J}_n(\mathbf{u}_n, \theta_{n,0}) = (\mathbf{u}_n, \theta_{n,0})$.

According to the above, to prove existence of solution to (3.27) in what follows we equivalently prove that $\mathcal{J}_n$ satisfies the hypotheses of Theorem 3.2.5. Before doing that, we must verify that $\mathcal{J}_n$ is well-defined. This is addressed in the following Lemma.

Lemma 3.2.6. *Let $\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$ with $\delta > 0$ satisfying*

$$\frac{C_D C_{O_\mathrm{T}}}{\alpha_\mathrm{F}\alpha_\mathrm{T}}(\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}) C_{\mathrm{lift},1}\, \delta^{1/12}\, \|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \le \frac{1}{4}, \tag{3.32}$$

and assume that the lifting θ_1 satisfies

$$\frac{C_D\, C_{O_\mathrm{T}}}{\alpha_\mathrm{F}\, \alpha_\mathrm{T}}\, (\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}})\, \|\theta_1\|_{0,\infty,\Omega} \;\le\; \frac{1}{4}. \tag{3.33}$$

Assume further that θ_D satisfies the following estimate

$$C_\mathbf{u}\, (\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}})\, C_{\mathrm{lift},2}\, \delta^{-3/4}(1+\delta^2)^{1/2}\, \|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \;\le\; \frac{2\mu\alpha_\mathrm{f}}{C_\mathrm{tr}^3\, C_{\mathrm{Sob},\Sigma}^2}, \tag{3.34}$$

where $C_\mathbf{u}$ is the positive constant defined in (3.28)*. Then, $\mathcal{J}_n(\mathbf{X}_n) \subseteq \mathbf{X}_n$ and for each $(\mathbf{w},\phi) \in \mathbf{X}_n$, there exists a unique $(\mathbf{u},\theta) \in \mathbf{X}_n$, such that $\mathcal{J}_n(\mathbf{w},\phi) = (\mathbf{u},\theta)$.*

Proof. Given $(\mathbf{w},\phi) = ((\mathbf{w}_\mathrm{f}, \mathbf{w}_\mathrm{m}), \phi) \in \mathbf{X}_n$, we first observe that (3.31) is an uncoupled system of linear equations. Thus, to prove that operator $\mathcal{J}_n$ is well-defined it suffices to prove the well-posedness of the two equations in (3.31) separately.

For the subsequent analysis, we let $\gamma_\mathbf{g} := \|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}$.

First, we use the well-known trace inequality with constant $C_\mathrm{tr} > 0$ (cf. (1.5)), and estimates (3.23b) and (3.34), to obtain

$$\begin{aligned} \|\mathbf{w}_\mathrm{f}\cdot\mathbf{n}\|_{0,\Sigma} \;&\le\; C_\mathrm{tr}\|\mathbf{w}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \;\le\; C_\mathrm{tr}\, C_\mathbf{u}\, \gamma_\mathbf{g} \|\theta_1\|_{1,\Omega} \\ &\le\; C_\mathrm{tr}\, C_\mathbf{u}\, \gamma_\mathbf{g}\, C_{\mathrm{lift},2}\, \delta^{-3/4}(1+\delta^2)^{1/2}\, \|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \\ &\le\; \frac{2\mu\alpha_\mathrm{f}}{C_\mathrm{tr}^2\, C_{\mathrm{Sob},\Sigma}^2}, \end{aligned} \tag{3.35}$$

which implies that $\mathbf{w}_{\mathrm{f}}$ satisfies (3.9). Then, thanks to Lemma 3.2.1 we have that $A_{\mathrm{F}}(\cdot,\cdot)+O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}};\cdot,\cdot)$ is elliptic on $\mathbf{V}_n$, which together with the Lax–Milgram Lemma implies that there exists a unique $\mathbf{u}\in\mathbf{V}_n$ solution to the first equation of (3.31).

Similarly, since $\Psi_n\subseteq\Psi_{\infty,0}$ and $\mathbf{V}_n\subseteq\mathbf{V}$, from (3.17) we have that $A_{\mathrm{T}}(\cdot,\cdot)+O_{\mathrm{T}}(\mathbf{w};\cdot,\cdot)$ is Ψ_n-elliptic. Then, owing to the Lax–Milgram Lemma we obtain that there exists a unique $\theta\in\Psi_n$, solution to the second equation of (3.31). According to the above, we have proved that there exists a unique $(\mathbf{u},\theta)\in\mathbf{V}_n\times\Psi_n$, such that $\mathcal{J}_n(\mathbf{w},\phi)=(\mathbf{u},\theta)$.

To conclude that $\mathcal{J}_n(\mathbf{X}_n)\subseteq\mathbf{X}_n$, it remains to prove that the aforementioned solution $(\mathbf{u},\theta)$ belongs to $\mathbf{X}_n$. To that end, we first notice that, since $\mathbf{u}$ satisfies the first equation in (3.31), from the ellipticity of $A_{\mathrm{F}}(\cdot,\cdot)+O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}};\cdot,\cdot)$, the continuity of D, the identity $C_{\mathbf{u}}=\alpha_{\mathrm{F}}^{-1}C_D(C_\theta+1)$ (cf. (3.28) and (3.29)) and the fact that $\phi\in\mathbf{X}_n$, we have

$$\begin{aligned}\|\mathbf{u}\|_{\mathbf{H}} &\le \alpha_{\mathrm{F}}^{-1}\left(C_D\,\|\phi\|_{1,\Omega}\,\gamma_{\mathbf{g}}+C_D\,\|\theta_1\|_{1,\Omega}\,\gamma_{\mathbf{g}}\right)\\ &\le \alpha_{\mathrm{F}}^{-1}\,C_D\,(C_\theta+1)\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega}\le C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega}.\end{aligned}\tag{3.36}$$

Similarly, since θ satisfies the second equation of (3.31), from the estimate above, the ellipticity of $A_{\mathrm{T}}(\cdot,\cdot)+O_{\mathrm{T}}(\mathbf{w};\cdot,\cdot)$ (cf. (3.17)), the continuity of A_{T} (cf. (3.5)) and O_{T} (cf. (3.13a)), and estimate (3.23a), it follows that

$$\begin{aligned}\|\theta\|_{1,\Omega} &\le\alpha_{\mathrm{T}}^{-1}\big[C_{A_{\mathrm{T}}}\|\theta_1\|_{1,\Omega}+C_{O_{\mathrm{T}}}\,\|\mathbf{u}\|_{\mathbf{H}}\,\big(C_{\mathrm{lift},1}\delta^{1/12}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}+\|\theta_1\|_{0,\infty,\Omega}\big)\big]\\ &\le\alpha_{\mathrm{T}}^{-1}\big[C_{A_{\mathrm{T}}}\|\theta_1\|_{1,\Omega}+C_{O_{\mathrm{T}}}C_{\mathbf{u}}\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega}\big(C_{\mathrm{lift},1}\delta^{1/12}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}+\|\theta_1\|_{0,\infty,\Omega}\big)\big].\end{aligned}\tag{3.37}$$

In addition, using the definition of $C_{\mathbf{u}}$ in (3.28), from (3.32) and (3.33), we obtain, respectively

$$\alpha_{\mathrm{T}}^{-1}\,C_{O_{\mathrm{T}}}\,C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega}\,C_{\mathrm{lift},1}\,\delta^{1/12}\,\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}\le\left(\frac{C_{A_{\mathrm{T}}}+\alpha_{\mathrm{T}}}{2\alpha_{\mathrm{T}}}\right)\|\theta_1\|_{1,\Omega},\tag{3.38}$$

and

$$\alpha_{\mathrm{T}}^{-1}\,C_{O_{\mathrm{T}}}\,C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega}\,\|\theta_1\|_{0,\infty,\Omega}\le\left(\frac{C_{A_{\mathrm{T}}}+\alpha_{\mathrm{T}}}{2\alpha_{\mathrm{T}}}\right)\|\theta_1\|_{1,\Omega}.\tag{3.39}$$

Therefore, by combining (3.37), (3.38), and (3.39), we obtain

$$\|\theta\|_{1,\Omega}\le C_\theta\|\theta_1\|_{1,\Omega}.\tag{3.40}$$

According to the above, $(\mathbf{u},\theta)\in\mathbf{X}_n$, thus $\mathcal{J}_n(\mathbf{X}_n)\subseteq\mathbf{X}_n$, which concludes the proof. □

Remark 3.2.1. *Observe that, according to* (3.22)*, the condition* (3.33) *can be replaced by*

$$\frac{C_D\, C_{O_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\, \alpha_{\mathrm{T}}} \left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}\right) \|\mathrm{E}(\theta_{\mathrm{D}})\|_{0,\infty,\Omega_\delta} \,\le\, \frac{1}{4},$$

which, in other words, means that the L^∞*-norm of the extension of the datum* θ_{D} *must be small enough on* $\Omega_\delta := \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \Gamma) < 2\delta\}$*, where* Ω_δ *is a small portion of the domain* Ω *near the boundary* Γ*. In particular, if* δ *is small enough so that* $\|\mathrm{E}(\theta_{\mathrm{D}})\|_{0,\infty,\Omega_\delta} \approx \|\theta_{\mathrm{D}}\|_{0,\infty,\Gamma}$*, one could simply assume that*

$$\frac{C_D\, C_{O_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\, \alpha_{\mathrm{T}}} \left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}\right) \|\theta_{\mathrm{D}}\|_{0,\infty,\Gamma} \,\le\, \frac{1}{4}.$$

Now, to apply the Brouwer's fixed-point theorem, we need to establish the continuity of operator $\mathcal{J}_n$. This result is established next.

Lemma 3.2.7. *If we assume that the hypotheses of Lemma 3.2.6 hold, then* $\mathcal{J}_n$ *is a continuous operator.*

Proof. Let $(\mathbf{w}, \phi) = ((\mathbf{w}_{\mathrm{f}}, \mathbf{w}_{\mathrm{m}}), \phi) \in \mathbf{X}_n$ and $\{(\mathbf{w}_j, \phi_j)\}_{j\in\mathbb{N}} = \{((\mathbf{w}_{\mathrm{f},j}, \mathbf{w}_{\mathrm{m},j}), \phi_j)\}_{j\in\mathbb{N}} \subseteq \mathbf{X}_n$, be such that $\|\mathbf{w}_j - \mathbf{w}\|_{\mathbf{H}} \overset{j\to\infty}{\longrightarrow} 0$ and $\|\phi_j - \phi\|_{1,\Omega} \overset{j\to\infty}{\longrightarrow} 0$, and let $\{(\mathbf{u}_j, \theta_j)\}_{j\in\mathbb{N}} = \{((\mathbf{u}_{\mathrm{f},j}, \mathbf{u}_{\mathrm{m},j}), \theta_j)\}_{j\in\mathbb{N}} \subseteq \mathbf{X}_n$ and $(\mathbf{u}, \theta) = ((\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}}), \theta) \in \mathbf{X}_n$ given, respectively, by

$$\mathcal{J}_n(\mathbf{w}_j, \phi_j) = (\mathbf{u}_j, \theta_j) \quad \forall\, j \in \mathbb{N} \quad \text{and} \quad \mathcal{J}_n(\mathbf{w}, \phi) = (\mathbf{u}, \theta).$$

To prove the continuity of $\mathcal{J}_n$ it suffices to prove that $\|\mathbf{u}_j - \mathbf{u}\|_{\mathbf{H}} \overset{j\to\infty}{\longrightarrow} 0$ and $\|\theta_j - \theta\|_{1,\Omega} \overset{j\to\infty}{\longrightarrow} 0$. To that end, given $j \in \mathbb{N}$, from (3.31) and the definition of $\mathcal{J}_n$ (cf. (3.30)), we first observe that

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u} - \mathbf{u}_j, \mathbf{v}) + O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}}; \mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) - O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f},j}; \mathbf{u}_{\mathrm{f},j}, \mathbf{v}_{\mathrm{f}}) &= D(\phi - \phi_j, \mathbf{v}) \\ A_{\mathrm{T}}(\theta - \theta_j, \psi) + O_{\mathrm{T}}(\mathbf{w}; \theta, \psi) - O_{\mathrm{T}}(\mathbf{w}_j; \theta_j, \psi) &= -O_{\mathrm{T}}(\mathbf{w} - \mathbf{w}_j; \theta_1, \psi), \end{aligned} \tag{3.41}$$

for all $\mathbf{v} = (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{V}_n$ and for all $\psi \in \Psi_n$. In turn, noticing that $\mathbf{w}_{\mathrm{f}}$ satisfies (3.35), we have that $A_{\mathrm{F}}(\cdot,\cdot) + O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}}; \cdot, \cdot)$ is elliptic (cf. (3.10)) on $\mathbf{V}_n$. Then, from (3.41) with $\mathbf{v} = \mathbf{u} - \mathbf{u}_j$, adding and subtracting $O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f},j}; \mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{f}} - \mathbf{u}_{\mathrm{f},j})$ and employing the continuity of O_{F} and D, and the fact that $\mathbf{w}_{\mathrm{f},j} \overset{j\to\infty}{\longrightarrow} \mathbf{w}_{\mathrm{f}}$ and $\phi_j \overset{j\to\infty}{\longrightarrow} \phi$, we arrive at

$$\alpha_{\mathrm{F}}\, \|\mathbf{u} - \mathbf{u}_j\|_{\mathbf{H}} \,\le\, C_{O_{\mathrm{F}}} \|\mathbf{w}_{\mathrm{f}} - \mathbf{w}_{\mathrm{f},j}\|_{1,\Omega_{\mathrm{f}}}\, \|\mathbf{u}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \,+\, C_D\, \|\phi - \phi_j\|_{1,\Omega}\, \gamma_{\mathbf{g}} \overset{j\to\infty}{\longrightarrow} 0, \tag{3.42}$$

where $\gamma_{\mathbf{g}} = \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$.

Similarly, in the second equation of (3.41), we let $\psi = \theta - \theta_j$, add and subtract $O_{\mathrm{T}}(\mathbf{w}_j;\theta,\theta-\theta_j)$, and employ the coercivity of $A_{\mathrm{T}}(\cdot,\cdot)+O_{\mathrm{T}}(\mathbf{w}_j;\cdot,\cdot)$ given in (3.17), the continuity of O_{T} (cf. (3.13b)), and the fact $\mathbf{w}_j \stackrel{j\to\infty}{\longrightarrow} \mathbf{w}$, to obtain

$$\alpha_{\mathrm{T}}\,\|\theta-\theta_j\|_{1,\Omega} \,\leq\, \widetilde{C}_{O_{\mathrm{T}}}\,\|\mathbf{w}-\mathbf{w}_j\|_{\mathbf{H}}\,(\|\theta+\theta_1\|_{1,\Omega_{\mathrm{f}}} + \|\theta+\theta_1\|_{0,\infty,\Omega_{\mathrm{m}}}) \stackrel{j\to\infty}{\longrightarrow} 0. \quad (3.43)$$

In this way, according to the definition of $\mathcal{J}_n$ (cf. (3.30)), from (3.42) and (3.43) we obtain that $\mathcal{J}_n(\mathbf{w}_j,\phi_j) \stackrel{j\to\infty}{\longrightarrow} \mathcal{J}_n(\mathbf{w},\phi)$, which implies the continuity of $\mathcal{J}_n$. □

Now we are in position of establishing the solvability result for the finite-dimensional nonlinear problem (3.27).

Theorem 3.2.8. *Let* $\delta > 0$ *satisfying* (3.32) *and let* $\theta_1 = \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega)$ *be such that* (3.33) *holds. Assume further that* (3.34) *holds. There exists at least one* $(\mathbf{u}_n,\theta_{n,0}) \in \mathbf{X}_n$ *solution to problem* (3.27).

Proof. The proof follows from Lemmas 3.2.6, 3.2.7 and Theorem 3.2.5. □

Now we address the solvability of the reduced problem (3.26). This result is established in the following Theorem.

Theorem 3.2.9. *Let* $\delta > 0$ *satisfying* (3.32) *and let* $\theta_1 = \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega)$ *be such that* (3.33) *holds. Assume further that* (3.34) *holds. Then, there exists at least one solution* $(\mathbf{u},\theta_0) \in \mathbf{V}\times\mathrm{H}^1_0(\Omega)$ *to problem* (3.26)*. In addition, the solution satisfies*

$$\|\mathbf{u}\|_{\mathbf{H}} \,\leq\, C_{\mathbf{u}}\,(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\,\|\theta_1\|_{1,\Omega}, \quad (3.44)$$

and

$$\|\theta_0\|_{1,\Omega} \,\leq\, C_\theta\,\|\theta_1\|_{1,\Omega}, \quad (3.45)$$

with $C_{\mathbf{u}}$ *and* C_θ*, being the positive constants defined in* (3.28) *and* (3.29)*, respectively.*

Proof. In what follow we proceed similarly to the proof of [7, Theorem 2.3]. To that end, for each $n\in\mathbb{N}$, we let $(\mathbf{u}_n,\theta_{n,0}) := ((\mathbf{u}_{n,\mathrm{f}},\mathbf{u}_{n,\mathrm{m}}),\theta_{n,0}) \in \mathbf{X}_n$ be a solution of problem (3.27) and let $\{(\mathbf{u}_n,\theta_{n,0})\}_{n\in\mathbb{N}} \subseteq \mathbf{V}\times\mathrm{H}^1_0(\Omega)$ be the resulting sequence. In turn, for a fixed $1\leq i\leq n$, we let $(\mathbf{z}_i,\varphi_i) := ((\mathbf{z}_{i,\mathrm{f}},\mathbf{z}_{i,\mathrm{m}}),\varphi_i) \in \mathbf{V}_n\times\Psi_n$ be the i-th basis function of $\mathbf{V}_n\times\Psi_n$.

First we notice that, since $(\mathbf{u}_n,\theta_{n,0}) \in \mathbf{X}_n$, then for all $n\in\mathbb{N}$,

$$\|\mathbf{u}_n\|_{\mathbf{H}} \,\leq\, C_{\mathbf{u}}\,(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}}+\|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_1\|_{1,\Omega} \quad \text{and} \quad \|\theta_{n,0}\|_{1,\Omega} \,\leq\, C_\theta\,\|\theta_1\|_{1,\Omega}, \quad (3.46)$$

thus $\{(\mathbf{u}_n, \theta_{n,0})\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of $\mathbf{V}\times \mathrm{H}^1_0(\Omega)$, which together with the fact that $\mathbf{V}$ is a closed subspace of $\mathbf{H}$, implies that there exists a subsequence, namely $\{(\widehat{\mathbf{u}}_n, \widehat{\theta}_{n,0})\}_{n\in\mathbb{N}} \subseteq \{(\mathbf{u}_n, \theta_{n,0})\}_{n\in\mathbb{N}}$, that weakly converges to some function $(\mathbf{u}, \theta_0) = ((\mathbf{u}_\mathrm{f}, \mathbf{u}_\mathrm{m}), \theta_0)$ in $\mathbf{V} \times \mathrm{H}^1_0(\Omega)$, that is,

$$\begin{aligned}&\widehat{\mathbf{u}}_n = (\widehat{\mathbf{u}}_{n,\mathrm{f}}, \widehat{\mathbf{u}}_{n,\mathrm{m}}) \overset{n\to\infty}{\rightharpoonup} \mathbf{u} = (\mathbf{u}_\mathrm{f}, \mathbf{u}_\mathrm{m}) \in \mathbf{V} \subseteq \mathbf{H} = \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}) \times \mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m}),\\ &\text{and} \quad \widehat{\theta}_{n,0} \overset{n\to\infty}{\rightharpoonup} \theta_0 \in \mathrm{H}^1_0(\Omega).\end{aligned} \tag{3.47}$$

In the sequel we prove that $(\mathbf{u}, \theta_0)$ is a solution to (3.26). In fact, from the second weak convergence in (3.47) we have that for each $i \in \mathbb{N}$, there holds

$$|A_\mathrm{T}(\widehat{\theta}_{n,0} - \theta_0, \varphi_i)| \le \max\{\kappa_\mathrm{f}, \kappa_\mathrm{m}\}|(\nabla(\widehat{\theta}_{n,0} - \theta_0), \nabla\varphi_i)_\Omega| \overset{n\to\infty}{\longrightarrow} 0,$$

thus

$$\lim_{n\to\infty} A_\mathrm{T}(\widehat{\theta}_{n,0}, \varphi_i) \;=\; A_\mathrm{T}(\theta_0, \varphi_i). \tag{3.48}$$

Now, recalling that $\varphi_i \in \Psi_{\infty,0}$ for all $i \in \mathbb{N}$, by applying the Green formula (3.12), we deduce that

$$\begin{aligned} O_\mathrm{T}(\widehat{\mathbf{u}}_n; \widehat{\theta}_{n,0} + \theta_1, \varphi_i) \;&=\; -\,O_\mathrm{T}(\widehat{\mathbf{u}}_n; \varphi_i, \widehat{\theta}_{n,0} + \theta_1)\\ &=\; -\,(\widehat{\mathbf{u}}_{n,\mathrm{f}}(\widehat{\theta}_{n,0} + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{f}} \;-\; (\widehat{\mathbf{u}}_{n,\mathrm{m}}(\widehat{\theta}_{n,0} + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{m}}. \end{aligned} \tag{3.49}$$

In turn, since $\{\widehat{\mathbf{u}}_{n,\mathrm{f}}\}_{n\in\mathbb{N}}$ converges weakly to $\mathbf{u}_\mathrm{f}$ in $\mathbf{H}^1(\Omega_\mathrm{f})$ and since $\mathbf{H}^1(\Omega_\mathrm{f})$ is compactly embedded in $\mathbf{L}^4(\Omega_\mathrm{f})$, it follows that $\{\widehat{\mathbf{u}}_{n,\mathrm{f}}\}_{n\in\mathbb{N}}$ converges strongly to $\mathbf{u}_\mathrm{f}$ in $\mathbf{L}^4(\Omega_\mathrm{f})$, and analogously we have that $\{\widehat{\theta}_{n,0}\}_{n\in\mathbb{N}}$ converges strongly to θ_0 in $\mathrm{L}^4(\Omega)$. These strong convergences and the fact that $\theta_1 \in \mathrm{W}^{1,4}(\Omega)$, imply that

$$\lim_{n\to\infty} (\widehat{\mathbf{u}}_{n,\mathrm{f}}(\widehat{\theta}_{n,0} + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{f}} = (\mathbf{u}_\mathrm{f}\,(\theta_0 + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{f}}. \tag{3.50}$$

On the other hand, the strong convergence of $\{\widehat{\theta}_{n,0}\}_{n\in\mathbb{N}}$ to θ_0 in $\mathrm{L}^4(\Omega)$ and the fact that $\nabla\varphi_i \in \mathbf{L}^4(\Omega)$ imply that $\{\widehat{\theta}_{n,0}\nabla\varphi_i\}_{n\in\mathbb{N}}$ convergences strongly to $\theta_0\nabla\varphi_i$ in $\mathbf{L}^2(\Omega)$. Then, similarly to (3.50), this strong convergence and the weak convergence of $\{\widehat{\mathbf{u}}_{n,\mathrm{m}}\}_{n\in\mathbb{N}}$ to $\mathbf{u}_\mathrm{m}$ in $\mathbf{L}^2(\Omega_\mathrm{m})$, imply that

$$\lim_{n\to\infty} (\widehat{\mathbf{u}}_{n,\mathrm{m}}(\widehat{\theta}_{n,0} + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{m}} = (\mathbf{u}_\mathrm{m}\,(\theta_0 + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{m}}. \tag{3.51}$$

In this way, from the second equation of (3.27), and from (3.48), (3.49), (3.50) and

(3.51), it follows that

$$
\begin{aligned}
-\,A_{\mathrm{T}}(\theta_1,\varphi_i) = & \lim_{n\to\infty}\Big[A_{\mathrm{T}}(\widehat{\theta}_{n,0},\varphi_i) \,+\, O_{\mathrm{T}}(\widehat{\mathbf{u}}_n;\widehat{\theta}_{n,0}+\theta_1,\varphi_i)\Big] \\
= & \lim_{n\to\infty}\Big[A_{\mathrm{T}}(\widehat{\theta}_{n,0},\varphi_i) \,-\, O_{\mathrm{T}}(\widehat{\mathbf{u}}_n;\varphi_i,\widehat{\theta}_{n,0}+\theta_1)\Big] \\
= & \, A_{\mathrm{T}}(\theta_0,\varphi_i) \,-\, O_{\mathrm{T}}(\mathbf{u};\varphi_i,\theta_0+\theta_1) \\
= & \, A_{\mathrm{T}}(\theta_0,\varphi_i) \,+\, O_{\mathrm{T}}(\mathbf{u};\theta_0+\theta_1,\varphi_i).
\end{aligned}
\tag{3.52}
$$

Analogously to the above, from the first equation of (3.27) and using the fact that $\{\widehat{\mathbf{u}}_{n,\mathrm{f}}\}_{n\in\mathbb{N}}$ and $\{\widehat{\theta}_{n,0}\}_{n\in\mathbb{N}}$ converge strongly to $\mathbf{u}$ in $\mathbf{L}^4(\Omega_{\mathrm{f}})$ and θ_0 in $\mathrm{L}^4(\Omega)$, respectively, we deduce that

$$
\begin{aligned}
D(\theta_1,\mathbf{z}_i) = & \lim_{n\to\infty}\Big[A_{\mathrm{F}}(\widehat{\mathbf{u}}_n,\mathbf{z}_i) \,+\, O_{\mathrm{F}}(\widehat{\mathbf{u}}_{n,\mathrm{f}};\widehat{\mathbf{u}}_{n,\mathrm{f}},\mathbf{z}_{i,\mathrm{f}}) \,-\, D(\widehat{\theta}_{n,0},\mathbf{z}_i)\Big] \\
= & \, A_{\mathrm{F}}(\mathbf{u},\mathbf{z}_i) \,+\, O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{z}_{i,\mathrm{f}}) \,-\, D(\theta_0,\mathbf{z}_i).
\end{aligned}
\tag{3.53}
$$

In this way, from (3.52), (3.53), and the fact that the basis $\{(\mathbf{z}_i,\varphi_i)\}_{i\in\mathbb{N}}$ is dense in $\mathbf{V}\times\mathrm{W}^{1,4}_0(\Omega)$, we obtain

$$
\begin{aligned}
A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) \,+\, O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) \,-\, D(\theta_0,\mathbf{v}) \;&=\; D(\theta_1,\mathbf{v}), \\
A_{\mathrm{T}}(\theta_0,\psi) \,+\, O_{\mathrm{T}}(\mathbf{u};\theta_0+\theta_1,\psi) \;&=\; -\,A_{\mathrm{T}}(\theta_1,\psi),
\end{aligned}
\tag{3.54}
$$

for all $\mathbf{v}\in\mathbf{V}$ and for all $\psi\in\mathrm{W}^{1,4}_0(\Omega)$. Since each term in the first and second equation of (3.54) defines a continuous linear functional on $\mathbf{V}$ and $\mathrm{W}^{1,4}_0(\Omega)$, respectively, particularly from the second equation of (3.54), we deduce (2.1d) and (2.2c) in the sense of distributions, that is

$$
-\kappa_{\mathrm{f}}\,\Delta\,\theta_{\mathrm{f}} \,+\, \mathbf{u}_{\mathrm{f}}\cdot\nabla\,\theta_{\mathrm{f}} = 0 \quad \text{and} \quad -\,\kappa_{\mathrm{m}}\,\Delta\,\theta_{\mathrm{m}} \,+\, \mathbf{u}_{\mathrm{m}}\cdot\nabla\,\theta_{\mathrm{m}} = 0.
$$

Hence, recalling that $\mathbf{u}_{\mathrm{f}}\cdot\nabla\theta_0|_{\Omega_{\mathrm{f}}}\in\mathrm{L}^2(\Omega_{\mathrm{f}})$ and $\mathbf{u}_{\mathrm{m}}\cdot\nabla\theta_0|_{\Omega_{\mathrm{m}}}\in\mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})'$, we observe that the second equation of (3.54) implies the identity

$$
A_{\mathrm{T}}(\theta_0,\psi)+(\mathbf{u}_{\mathrm{f}}\cdot\nabla\theta_0,\psi)_{\Omega_{\mathrm{f}}}+\langle\mathbf{u}_{\mathrm{m}}\cdot\nabla\theta_0,\psi\rangle_{\mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})',\mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})}+O_{\mathrm{T}}(\mathbf{u};\theta_1,\psi) = -A_{\mathrm{T}}(\theta_1,\psi),
$$

for all $\psi\in\mathrm{H}^1_0(\Omega)$. Therefore, since $\mathbf{u}_{\mathrm{m}}\cdot\nabla\theta_0|_{\Omega_{\mathrm{m}}}$ also belongs to $\mathrm{L}^1(\Omega_{\mathrm{m}})$, from the latter we can recover the equation

$$
A_{\mathrm{T}}(\theta_0,\psi) \,+\, O_{\mathrm{T}}(\mathbf{u};\theta_0+\theta_1,\psi) = -\,A_{\mathrm{T}}(\theta_1,\psi) \quad \forall\,\psi\in\Psi_{\infty,0},
$$

thus, $(\mathbf{u},\theta_0)$ satisfies (3.26), which concludes the proof of existence of solution.

Regarding estimates (3.44) and (3.45), analogously to the proof of [63, Theorem 2.2] we observe that these estimates follow from (3.46) and the fact that the norm is a lower semicontinuous function for the weak topology (see [16, Proposition 3.5]). In fact, using (3.46) and the aforementioned property of the norm, we obtain

$$\|\mathbf{u}\|_{\mathbf{H}} \leq \liminf_{n\to\infty} \|\widehat{\mathbf{u}}_n\|_{\mathbf{H}} \ \leq\ C_{\mathbf{u}}\left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}\right)\|\theta_1\|_{1,\Omega},$$

and

$$\|\theta_0\|_{1,\Omega} \leq \liminf_{n\to\infty} \|\widehat{\theta}_{n,0}\|_{1,\Omega} \ \leq\ C_\theta\, \|\theta_1\|_{1,\Omega},$$

which concludes the proof. □

We end this section by deriving the corresponding estimate for the pressure p and the Lagrange multiplier λ.

Corollary 3.2.10. *Assume that the hypotheses of Lemma 3.2.6 hold, let* $(\mathbf{u},\theta_0) = ((\mathbf{u}_{\mathrm{f}},\mathbf{u}_{\mathrm{m}}),\theta_0) \in \mathbf{V}\times \mathrm{H}^1_0(\Omega)$ *be a solution of* (3.26) *and let* $(p,\lambda)\in \mathrm{Q}$ *be such that* $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1) \in \mathbf{H}\times\mathrm{Q}\times\mathrm{H}^1(\Omega)$ *is a solution to* (3.2)*. There exist* C_1*,* $C_2 > 0$*, such that*

$$\|(p,\lambda)\|_{\mathrm{Q}} \leq (C_1 + C_2\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega})\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega},$$

with $\gamma_{\mathbf{g}} = \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$.

Proof. The result is a direct consequence of the inf-sup condition (3.11) and the first equation of (3.2). In fact, it is easy to see that

$$\begin{aligned}
\beta\|(p,\lambda)\|_{\mathrm{Q}} &\leq \sup_{\mathbf{v}\in\mathbf{H}\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(p,\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} \\
&= \sup_{\mathbf{v}\in\mathbf{H}\setminus\{\mathbf{0}\}} \frac{-A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) \,-\, O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) \,+\, D(\theta,\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}}} \\
&\leq \big(C_{A_{\mathrm{F}}}\,\|\mathbf{u}\|_{\mathbf{H}} \,+\, C_{O_{\mathrm{F}}}\,\|\mathbf{u}_{\mathrm{f}}\|^2_{1,\Omega_{\mathrm{f}}} \,+\, C_D\gamma_{\mathbf{g}}\,\|\theta\|_{1,\Omega}\big) \\
&\leq (C_{A_{\mathrm{F}}} + C_{O_{\mathrm{F}}}C_{\mathbf{u}}\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega})C_{\mathbf{u}}\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega} + C_D\,\gamma_{\mathbf{g}}(C_\theta+1)\|\theta_1\|_{1,\Omega},
\end{aligned}$$

which implies the result. □

3.2.3 Uniqueness of solution

Having established existence and stability of problem (3.2), under suitable assumptions on data, now we address the uniqueness of solution. This is established in the following theorem.

Theorem 3.2.11. *Assume that the hypotheses of Lemma 3.2.6 hold and let* $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1) \in \mathbf{H}\times\mathrm{Q}\times\mathrm{H}^1(\Omega)$ *be a solution to* (3.2)*, with*

$\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$ *and* $\theta_0 \in \mathrm{H}^1_0(\Omega)$. *Assume further that* $\theta_0|_{\Omega_\mathrm{m}} \in \mathrm{W}^{1,4}(\Omega_\mathrm{m})$ *and that*

$$\begin{aligned}\Big[(C_1\,\gamma_\mathbf{g} + C_2)\, C_{\mathrm{lift},2}\delta^{-3/4}(1+\delta^2)^{1/2} + C_3 C_{\mathrm{lift},3}(1+\delta^{-2})^{1/2}\Big] \|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \\ + C_3\|\theta_0\|_{1,4,\Omega_\mathrm{m}} + C_4\gamma_\mathbf{g} \,<\, 1,\end{aligned} \tag{3.55}$$

with $\gamma_\mathbf{g} = \|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}$ *and* C_1*,* C_2*,* C_3 *and* C_4 *the positive constants given in* (3.60)*. Then the solution of problem* (3.2) *is unique.*

Proof. Let $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1) \in \mathbf{H}\times \mathrm{Q}\times \mathrm{H}^1(\Omega)$ and $(\overline{\mathbf{u}},(\overline{p},\overline{\lambda}),\overline{\theta}) = (\overline{\mathbf{u}},(\overline{p},\overline{\lambda}),\overline{\theta}_0+\theta_1) \in \mathbf{H}\times \mathrm{Q}\times \mathrm{H}^1(\Omega)$ be two solutions of problem (3.2). It follows that $(\mathbf{u},\theta_0), (\overline{\mathbf{u}},\overline{\theta}_0) \in \mathbf{V}\times \mathrm{H}^1_0(\Omega)$ are solutions of (3.26), which implies

$$\begin{aligned} A_\mathrm{F}(\mathbf{u}-\overline{\mathbf{u}},\mathbf{v}) + O_\mathrm{F}(\mathbf{u}_\mathrm{f};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) - O_\mathrm{F}(\overline{\mathbf{u}}_\mathrm{f};\overline{\mathbf{u}}_\mathrm{f},\mathbf{v}_\mathrm{f}) - D(\theta_0-\overline{\theta}_0,\mathbf{v}) &= 0,\\ A_\mathrm{T}(\theta_0-\overline{\theta}_0,\psi) + O_\mathrm{T}(\mathbf{u};\theta_0,\psi) - O_\mathrm{T}(\overline{\mathbf{u}};\overline{\theta}_0,\psi) + O_\mathrm{T}(\mathbf{u}-\overline{\mathbf{u}};\theta_1,\psi) &= 0,\end{aligned} \tag{3.56}$$

for all $\mathbf{v}\in\mathbf{V}$ and for all $\psi \in \Psi_{\infty,0}$. From the first equation of (3.56) we observe that by adding and subtracting $O_\mathrm{F}(\mathbf{u}_\mathrm{f};\overline{\mathbf{u}}_\mathrm{f},\mathbf{v}_\mathrm{f})$, taking $\mathbf{v} = \mathbf{u}-\overline{\mathbf{u}}$, employing the coercivity of $A_\mathrm{F}(\cdot,\cdot) + O_\mathrm{F}(\mathbf{u};\cdot,\cdot)$ (cf. (3.10)) , and the continuity of O_F and D, we have

$$\alpha_\mathrm{F}\,\|\mathbf{u}-\overline{\mathbf{u}}\|_\mathbf{H} \,\le\, C_{O_\mathrm{F}}\|\mathbf{u}_\mathrm{f}-\overline{\mathbf{u}}_\mathrm{f}\|_{1,\Omega_\mathrm{f}}\,\|\overline{\mathbf{u}}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \,+\, C_D\,\gamma_\mathbf{g}\,\|\theta_0-\overline{\theta}_0\|_{1,\Omega}. \tag{3.57}$$

In turn, in the second equation of (3.56) we add and subtract $O_\mathrm{T}(\mathbf{u};\overline{\theta}_0,\psi)$, recall the fact that $\theta_0|_{\Omega_\mathrm{m}}, \overline{\theta}_0|_{\Omega_\mathrm{m}} \in \mathrm{W}^{1,4}(\Omega_\mathrm{m}) \subseteq \mathrm{L}^\infty(\Omega_\mathrm{m})$ to define $\psi = \theta_0-\overline{\theta}_0 \in \Psi_{\infty,0}$, and then employ the coercivity of $A_\mathrm{T}(\cdot,\cdot)+O_\mathrm{T}(\mathbf{u};\cdot,\cdot)$ (cf. (3.17)) and estimate (3.14), to get

$$\alpha_\mathrm{T}\,\|\theta_0-\overline{\theta}_0\|_{1,\Omega} \,\le\, \overline{C}_{O_\mathrm{T}}\,\|\mathbf{u}-\overline{\mathbf{u}}\|_\mathbf{H}\left(\|\overline{\theta}_0+\theta_1\|_{1,\Omega_\mathrm{f}} \,+\, \|\overline{\theta}_0+\theta_1\|_{1,4,\Omega_\mathrm{m}}\right). \tag{3.58}$$

Then, summing up (3.57) and (3.58), we arrive at

$$\begin{aligned}\|\mathbf{u}-\overline{\mathbf{u}}\|_\mathbf{H} + \|\theta_0-\overline{\theta}_0\|_{1,\Omega} \le \alpha_\mathrm{F}^{-1}\,\left[C_{O_\mathrm{F}}\|\mathbf{u}_\mathrm{f}-\overline{\mathbf{u}}_\mathrm{f}\|_{1,\Omega_\mathrm{f}}\,\|\overline{\mathbf{u}}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \,+\, C_D\,\gamma_\mathbf{g}\,\|\theta_0-\overline{\theta}_0\|_{1,\Omega}\right]\\ + \alpha_\mathrm{T}^{-1}\,\left[\overline{C}_{O_\mathrm{T}}\,\|\mathbf{u}-\overline{\mathbf{u}}\|_\mathbf{H}\left(\|\overline{\theta}_0+\theta_1\|_{1,\Omega_\mathrm{f}} \,+\, \|\overline{\theta}_0+\theta_1\|_{1,4,\Omega_\mathrm{m}}\right)\right],\end{aligned}$$

and then, using the fact that $\|\overline{\mathbf{u}}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \le \|\overline{\mathbf{u}}\|_\mathbf{H} \le C_\mathbf{u}\gamma_\mathbf{g}\|\theta_1\|_{1,\Omega}$ (cf. (3.36)) and $\|\overline{\theta}_0+\theta_1\|_{1,\Omega_\mathrm{f}} \le \|\overline{\theta}_0+\theta_1\|_{1,\Omega} \le (C_\theta+1)\|\theta_1\|_{1,\Omega}$ (cf. (3.40)), we obtain

$$\begin{aligned}\|\mathbf{u}-\overline{\mathbf{u}}\|_\mathbf{H} + \|\theta_0-\overline{\theta}_0\|_{1,\Omega} \le (C_1\gamma_\mathbf{g}+C_2)\|\theta_1\|_{1,\Omega}\|\mathbf{u}-\overline{\mathbf{u}}\|_\mathbf{H} + C_3\|\overline{\theta}_0\|_{1,4,\Omega_\mathrm{m}}\|\mathbf{u}-\overline{\mathbf{u}}\|_\mathbf{H}\\ + C_3\|\theta_1\|_{1,4,\Omega_\mathrm{m}}\|\mathbf{u}-\overline{\mathbf{u}}\|_\mathbf{H} + C_4\gamma_\mathbf{g}\|\theta_0-\overline{\theta}_0\|_{1,\Omega},\end{aligned} \tag{3.59}$$

where

$$
\begin{aligned}
C_1 &:= \alpha_{\mathrm{F}}^{-1}\, C_{O_{\mathrm{F}}}\, C_{\mathbf{u}}, \\
C_2 &:= \alpha_{\mathrm{T}}^{-1}\, \overline{C}_{O_{\mathrm{T}}}\, \left(C_\theta + 1\right), \\
C_3 &:= \alpha_{\mathrm{T}}^{-1}\, \overline{C}_{O_{\mathrm{T}}}, \\
C_4 &:= \alpha_{\mathrm{F}}^{-1}\, C_D.
\end{aligned}
\tag{3.60}
$$

In this way, recalling that estimates (3.23b) and (3.23c) imply

$$\|\theta_1\|_{1,\Omega} \le C_{\mathrm{lift},2}\, \delta^{-3/4}(1+\delta^2)^{1/2}\, \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma},$$

and

$$\|\theta_1\|_{1,4,\Omega} \le C_{\mathrm{lift},3}\, (1+\delta^{-2})^{1/2}\, \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma},$$

from (3.59) and (3.55) we readily obtain that $\mathbf{u} = \overline{\mathbf{u}}$ and $\theta_0 = \overline{\theta}_0$. Now, for the pressure and the Lagrange multiplier, from the inf-sup condition (3.11) we have that

$$
\begin{aligned}
&\beta\, \|(p-\overline{p}, \lambda - \overline{\lambda})\|_{\mathrm{Q}} \\
&\le \sup_{\mathbf{v}\in\mathbf{H}\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v}, (p-\overline{p}, \lambda-\overline{\lambda}))}{\|\mathbf{v}\|_{\mathbf{H}}} \\
&= \sup_{\mathbf{v}\in\mathbf{H}\setminus\{\mathbf{0}\}} \frac{-A_{\mathrm{F}}(\mathbf{u}-\overline{\mathbf{u}}, \mathbf{v}) - O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}}; \mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) + O_{\mathrm{F}}(\overline{\mathbf{u}}_{\mathrm{f}}; \overline{\mathbf{u}}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) + D(\theta_0 - \overline{\theta}_0, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}}},
\end{aligned}
$$

which after simple computations implies that $\|(p-\overline{p}, \lambda-\overline{\lambda})\|_{\mathrm{Q}} \le 0$, thus $p = \overline{p}$ and $\lambda = \overline{\lambda}$, which concludes the proof. □

3.3 Galerkin scheme

In this section we introduce and analyze a finite element scheme to approximate the solution of problem (3.2). We start by introducing the Galerkin scheme and reviewing the discrete stability properties of the forms involved. As we shall see next in the forthcoming sections, the analysis of the associated discrete scheme is analogous to the analysis of the finite-dimensional problem (3.31), employed to study the continuous problem (3.26).

3.3.1 Discrete problem

Let $\mathcal{T}_h^{\mathrm{f}}$ and $\mathcal{T}_h^{\mathrm{m}}$ be the respective regular triangulations of the domains Ω_{f} and Ω_{m} formed by shape-regular triangles T in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$ of diameter h_T and denote by h_{f} and h_{m} their corresponding mesh sizes. Assume that they match on Σ so that $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{f}} \cup \mathcal{T}_h^{\mathrm{m}}$ is a triangulation of $\Omega := \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$. Hereafter $h := \max\{h_{\mathrm{f}}, h_{\mathrm{m}}\}$.

Given an integer $l \geq 0$, for each $T \in \mathcal{T}_h$, we let $\mathrm{P}_l(T)$ be the space of polynomials functions on T of degree equal or less than l. Moreover, for each $T \in \mathcal{T}_h^{\mathrm{f}}$, we denote by $\mathbf{BR}(T)$ the local Bernardi–Raugel space (see [11, 62]),

$$\mathbf{BR}(T) := [\mathrm{P}_1(T)]^d \oplus \mathrm{span}\{\mathbf{p}_i \,:\, 1 \leq i \leq d+1\},$$

with $\mathbf{p}_i = \mathbf{n}_i \displaystyle\prod_{j=1, j\neq i}^{d+1} \eta_j$, where $\{\eta_1, \ldots, \eta_{d+1}\}$ are the baricentric coordinates of T, $\{\mathbf{n}_1, \ldots, \mathbf{n}_{d+1}\}$ are the unit outward normals to the opposite faces of the corresponding vertices of T, and $d \in \{2, 3\}$. In turn, for each $T \in \mathcal{T}_h^{\mathrm{m}}$ we consider the local Raviart–Thomas space of the lowest order (see [91])

$$\mathbf{RT}_0(T) := [\mathrm{P}_0(T)]^d \oplus \mathrm{P}_0(T)\mathbf{x},$$

where $\mathbf{x} := (x_1, \ldots, x_d)^t$ is a generic vector of $\mathbb{R}^d$. Hence, we define the following finite element subspaces:

$$\begin{aligned}
&\mathbf{H}_h(\Omega_{\mathrm{f}}) := \{\mathbf{v}_{\mathrm{f}} \in \mathbf{H}^1(\Omega_{\mathrm{f}}) : \mathbf{v}_{\mathrm{f}}|_T \in \mathbf{BR}(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{f}}\},\\
&\mathbf{H}_h(\Omega_{\mathrm{m}}) := \{\mathbf{v}_{\mathrm{m}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{m}}) : \mathbf{v}_{\mathrm{m}}|_T \in \mathbf{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{m}}\},\\
&\mathrm{L}_h(\Omega) := \{q \in \mathrm{L}^2(\Omega) : q|_T \in \mathrm{P}_0(T) \quad \forall\, T \in \mathcal{T}_h\},\\
&\Psi_h := \{\psi \in \mathrm{H}^1(\Omega) : \psi|_T \in \mathrm{P}_1(T) \quad \forall\, T \in \mathcal{T}_h\},\\
&\mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) := \mathbf{H}_h(\Omega_{\mathrm{f}}) \cap \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}),\\
&\mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}}) := \mathbf{H}_h(\Omega_{\mathrm{m}}) \cap \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}}),\\
&\mathrm{L}_{h,0}(\Omega) := \mathrm{L}_h(\Omega) \cap \mathrm{L}^2_0(\Omega),\\
&\Psi_{h,0} := \Psi_h \cap \mathrm{H}^1_0(\Omega).
\end{aligned}$$

It remains to introduce the finite element subspace for $\mathrm{H}^{1/2}(\Sigma)$. To do that, let us denote by Σ_h the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{f}}$ (or $\mathcal{T}_h^{\mathrm{m}}$), formed by faces F (edges in 2D and triangles in 3D) of diameter h_F, define $h_\Sigma := \max\{h_F : F \in \Sigma_h\}$ and consider the following subspace of $\mathrm{L}^2(\Sigma)$:

$$\Phi_h(\Sigma) := \Big\{ \phi_h : \Sigma \to \mathbb{R} : \quad \phi_h|_F \in \mathrm{P}_0(F), \quad \forall\, F \in \Sigma_h \Big\}. \tag{3.61}$$

To maintain clarity, we address the 2D case first, followed by the 3D case.

A subspace of $\mathrm{H}^{1/2}(\Sigma)$ **in 2D**:

Let us assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. If the number of edges of Σ_h is odd, we simply reduce it to the even case by adding one node to the discretization of the interface and locally modify the triangulation to keep the mesh conformity and regularity. According to the above, we define the following finite element subspace for $\mathrm{H}^{1/2}(\Sigma)$

$$\Lambda_h(\Sigma) := \{\xi_h \in \mathrm{C}^0(\Sigma) : \xi_h|_F \in \mathrm{P}_1(F) \quad \forall\, F \in \Sigma_{2h}\}.$$

A subspace of $\mathrm{H}^{1/2}(\Sigma)$ in 3D:

Let us now introduce an independent triangulation $\Sigma_{\widehat{h}}$ of Σ, by faces (triangles) F of diameter $\widehat{h}_F$, and define $\widehat{h}_\Sigma := \max\{\widehat{h}_F : F \in \Sigma_{\widehat{h}}\}$. Then, we define

$$\Lambda_h(\Sigma) \;=\; \Big\{\, \xi_h \,\in\, \mathrm{C}^0(\Sigma) : \quad \xi_h|_F \,\in\, \mathrm{P}_1(F) \quad \forall\, F \in \Sigma_{\widehat{h}} \,\Big\}\,. \tag{3.62}$$

As we shall see next in the proof of Lemma 3.3.3, the discrete counterpart of the inf-sup condition of the bilinear form B (cf. (3.11)) relies on the following inf-sup condition for the pair of subspaces $(\Phi_h(\Sigma), \Lambda_h(\Sigma))$:

$$\sup_{\substack{\phi_h \,\in\, \Phi_h(\Sigma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \;\geq\; \beta_\Sigma\, \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \,\in\, \Lambda_h(\Sigma)\,, \tag{3.63}$$

with $\beta_\Sigma > 0$, independent of h. In [58, Lemma 5.2], it has been established that if $d = 2$, estimate (3.63) is satisfied without additional assumptions. However, for the 3D case, it can be shown that if Σ_h is quasi-uniform and the pair $(h_\Sigma, \widehat{h}_\Sigma)$ satisfies

$$h_\Sigma \leq C_0 \widehat{h}_\Sigma, \quad \text{with} \quad C_0 \in (0,1), \tag{3.64}$$

(the existence of such a C_0 is detailed in the second part of the proof of [55, Lemma 7.5]), then estimate (3.63) holds. It's worth noting that this constraint on the mesh size explains the need of introducing the independent partition $\Sigma_{\widehat{h}}$ of Σ for the case $d = 3$. According to this, from now on we assume that for the 3D case, Σ_h is quasi-uniform and estimate (3.64) holds.

Now, we let $\mathrm{I}_h : \mathrm{C}^0(\overline{\Omega}) \to \Psi_h$ be the well-known Lagrange interpolation operator and recall that, under the assumption $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ and for a given $\delta > 0$, $\mathrm{E}_\delta(\theta_{\mathrm{D}})$ belongs to $\mathrm{W}^{1,4}(\Omega) \subseteq \mathrm{C}^0(\overline{\Omega})$ (cf. (1.6)). For a fixed $\delta > 0$ (to be specified below), we define the following approximation to θ_{D}:

$$\theta^\delta_{\mathrm{D},h} = \mathrm{I}_h(\mathrm{E}_\delta(\theta_{\mathrm{D}}))|_\Gamma \in \{\psi_{\mathrm{D},h} \in \mathrm{C}^0(\Gamma) : \psi_{\mathrm{D},h}|_F \in \mathrm{P}_1(F) \text{ for all } F \in \mathcal{E}_\Gamma\}, \tag{3.65}$$

where $\mathcal{E}_\Gamma$ stands for the set of edges/faces on Γ.

Let us observe that since Ω is a polyhedral domain, Ω_δ is also a polyhedron that can be discretized by shaped-regular elements. According to this, for the forthcoming analysis we let $\mathcal{T}_h^\delta$ be a triangulation of Ω_δ and assume that $\mathcal{T}_h^\delta \subseteq \mathcal{T}_h$.

In this way, defining the global spaces

$$\mathbf{H}_h := \mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f}) \times \mathbf{H}_{h,\Gamma_\mathrm{m}}(\Omega_\mathrm{m}) \quad \text{and} \quad \mathrm{Q}_h := \mathrm{L}_{h,0}(\Omega) \times \Lambda_h(\Sigma),$$

the Galerkin scheme associated to (3.2) reads:

Find $\mathbf{u}_h := (\mathbf{u}_{h,\mathrm{f}}, \mathbf{u}_{h,\mathrm{m}}) \in \mathbf{H}_h$, $(p_h, \lambda_h) \in \mathrm{Q}_h$ and $\theta_h \in \Psi_h$, such that $\theta_h|_\Gamma = \theta^\delta_{\mathrm{D},h}$, and

$$\begin{aligned} A_\mathrm{F}(\mathbf{u}_h, \mathbf{v}_h) \,+\, O^h_\mathrm{F}(\mathbf{u}_{h,\mathrm{f}}; \mathbf{u}_{h,\mathrm{f}}, \mathbf{v}_{h,\mathrm{f}}) \,+\, B(\mathbf{v}_h, (p_h, \lambda_h)) \,-\, D(\theta_h, \mathbf{v}_h) &= 0, \\ B(\mathbf{u}_h, (q_h, \xi_h)) &= 0, \\ A_\mathrm{T}(\theta_h, \psi_h) \,+\, O^h_\mathrm{T}(\mathbf{u}_h; \theta_h, \psi_h) &= 0, \end{aligned} \tag{3.66}$$

for all $\mathbf{v}_h \in \mathbf{H}_h$, for all $(q_h, \xi_h) \in \mathrm{Q}_h$ and for all $\psi_h \in \Psi_{h,0}$, where A_F, B, D, and A_T are the form defined in Section 3.2.1, while O^h_F and O^h_T are the skew-symmetric convection forms (see [99]), defined by

$$O^h_\mathrm{F}(\mathbf{w}_\mathrm{f}; \mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f}) \,:=\, ((\mathbf{w}_\mathrm{f} \cdot \nabla)\mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f})_{\Omega_\mathrm{f}} \,+\, \frac{1}{2}\,(\mathrm{div}\,\mathbf{w}_\mathrm{f}, \mathbf{u}_\mathrm{f} \cdot \mathbf{v}_\mathrm{f})_{\Omega_\mathrm{f}},$$

for all $\mathbf{w}_\mathrm{f}, \mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f} \in \mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, and

$$O^h_\mathrm{T}(\mathbf{w}; \theta, \psi) \,:=\, (\mathbf{w}_\mathrm{f} \cdot \nabla\theta, \psi)_{\Omega_\mathrm{f}} \,+\, (\mathbf{w}_\mathrm{m} \cdot \nabla\theta, \psi)_{\Omega_\mathrm{m}} \,+\, \frac{1}{2}\,(\mathrm{div}\,\mathbf{w}_\mathrm{f}, \theta\,\psi)_{\Omega_\mathrm{f}},$$

for all $\mathbf{w} = (\mathbf{w}_\mathrm{f}, \mathbf{w}_\mathrm{m}) \in \mathbf{H}_h$ and for all $\theta, \psi \in \Psi_h$. The motivation for this choice is given later on in Remark 3.3.1.

3.3.2 Existence of solution of the discrete scheme

In what follows we prove that the discrete problem (3.66) has at least one solution under suitable assumptions on the data. We begin the discussion by establishing the stability properties of the forms involved restricted to the corresponding discrete spaces.

Discrete stability properties

We begin by observing that the forms A_F, B, D, and A_T are continuous with the same constants described in Section 3.2.2 (see (3.4)-(3.6)). In turn, by using the Sobolev inequality (1.7) with $p = 4$, it is easy to see that

$$|O^h_\mathrm{F}(\mathbf{w}_\mathrm{f}; \mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f})| \,\leq\, \widehat{C}_{O_\mathrm{F}}\, \|\mathbf{w}_\mathrm{f}\|_{1,\Omega_\mathrm{f}}\, \|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}}\, \|\mathbf{v}_\mathrm{f}\|_{1,\Omega_\mathrm{f}}, \tag{3.67}$$

for all $\mathbf{w}_\mathrm{f}$, $\mathbf{u}_\mathrm{f}$, $\mathbf{v}_\mathrm{f} \in \mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$ with $\widehat{C}_{O_\mathrm{F}} := C^2_{\mathrm{Sob},\Omega_\mathrm{f}}\left(1 + \frac{\sqrt{d}}{2}\right)$. Furthermore, we observe that integrating by parts, there holds

$$O^h_\mathrm{F}(\mathbf{w}_\mathrm{f}; \mathbf{v}_\mathrm{f}, \mathbf{v}_\mathrm{f}) \,=\, \frac{1}{2}\left\langle \mathbf{w}_\mathrm{f} \cdot \mathbf{n}, |\mathbf{v}_\mathrm{f}|^2 \right\rangle_\Sigma \qquad \forall\, \mathbf{w}_\mathrm{f}, \mathbf{v}_\mathrm{f} \in \mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f}).$$

Now, let $\mathbf{V}_h$ be the discrete kernel of B, that is

$$\mathbf{V}_h := \{\mathbf{v} \in \mathbf{H}_h : B(\mathbf{v}, (q, \xi)) \,=\, 0 \qquad \forall\, (q, \xi) \in \mathrm{Q}_h\}. \tag{3.68}$$

Similarly to the continuous case, $\mathbf{v} = (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{V}_h$ if and only if

$$(q, \operatorname{div} \mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} \,+\, (q, \operatorname{div} \mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}} \,=\, 0 \quad \forall\, q \in \mathrm{L}_{h,0}(\Omega),$$

and

$$\langle \mathbf{v}_{\mathrm{f}} \cdot \mathbf{n} - \mathbf{v}_{\mathrm{m}} \cdot \mathbf{n}, \xi \rangle_{\Sigma} \,=\, 0 \quad \forall\, \xi \in \Lambda_h(\Sigma),$$

which imply that

$$(\operatorname{div} \mathbf{v}_{\mathrm{f}}, q)_{\Omega_{\mathrm{f}}} \,=\, 0 \quad \forall\, q \in \mathrm{L}_h(\Omega_{\mathrm{f}}) \quad \text{and} \quad \operatorname{div} \mathbf{v}_{\mathrm{m}} \,=\, 0 \quad \text{in} \quad \Omega_{\mathrm{m}},$$

where $\mathrm{L}_h(\Omega_{\mathrm{f}})$ is the set of functions of $\mathrm{L}_h(\Omega)$ restricted to Ω_{f}, more precisely,

$$\mathrm{L}_h(\Omega_{\mathrm{f}}) := \{q \in \mathrm{L}^2(\Omega_{\mathrm{f}}) : q|_T \in \mathrm{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{f}}\}.$$

According to the above, we observe that the discrete kernel (3.68) can be written as

$$\mathbf{V}_h := \{\mathbf{v} = (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{V}_{h,\mathrm{f}} \times \mathbf{V}_{h,\mathrm{m}} : \langle \mathbf{v}_{\mathrm{f}} \cdot \mathbf{n} - \mathbf{v}_{\mathrm{m}} \cdot \mathbf{n}, \xi \rangle_{\Sigma} = 0 \quad \forall\, \xi \in \Lambda_h(\Sigma)\}, \tag{3.69}$$

where

$$\begin{aligned} \mathbf{V}_{h,\mathrm{f}} :=& \quad \{\mathbf{v}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) : (q, \operatorname{div} \mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} = 0 \quad \forall\, q \in \mathrm{L}_h(\Omega_{\mathrm{f}})\}, \\ \mathbf{V}_{h,\mathrm{m}} :=& \quad \{\mathbf{v}_{\mathrm{m}} \in \mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}}) : \operatorname{div} \mathbf{v}_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}\}. \end{aligned}$$

Remark 3.3.1. *We observe here that if* $\mathbf{v} := (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{V}_h$*, then* $\mathbf{v}_{\mathrm{f}}$ *is not necessarily divergence-free, which motivates the introduction of the convective forms* O_{F}^h *and* O_{T}^h*.*

Let us observe now that the forms $a_{\mathrm{F,f}}$, $a_{\mathrm{F,m}}$, and A_{T}, are elliptic with the same constants of the continuous case (see (3.8) and (3.16)), that is,

$$\begin{aligned} a_{\mathrm{F,f}}(\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) \,\geq\, 2\mu\alpha_{\mathrm{f}} \,\|\mathbf{v}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}^2 \qquad & \forall\, \mathbf{v}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}), \\ a_{\mathrm{F,m}}(\mathbf{v}_{\mathrm{m}}, \mathbf{v}_{\mathrm{m}}) \,\geq\, C_{\mathbf{K}} \,\|\mathbf{v}_{\mathrm{m}}\|_{\operatorname{div};\Omega_{\mathrm{m}}}^2 \qquad & \forall\, \mathbf{v}_{\mathrm{m}} \in \mathbf{V}_{h,\mathrm{m}}, \\ A_{\mathrm{T}}(\psi, \psi) \,\geq\, \alpha_{\mathrm{T}} \,\|\psi\|_{1,\Omega}^2 \qquad & \forall\, \psi \in \Psi_{h,0}. \end{aligned} \tag{3.70}$$

In particular, using the ellipticity of $a_{\mathrm{F,f}}$ and $a_{\mathrm{F,m}}$ one can deduce that the form $A_{\mathrm{F}}(\cdot,\cdot) + O_{\mathrm{F}}^h(\mathbf{w}_{\mathrm{f}}; \cdot, \cdot)$, is elliptic on $\mathbf{V}_h$ for suitable $\mathbf{w}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$. More precisely, we have the following discrete version of Lemma 3.2.1. For its proof we refer the reader to [45, Lemma 10].

Lemma 3.3.1. *Let* $\mathbf{w}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$*, be such that*

$$\|\mathbf{w}_{\mathrm{f}} \cdot \mathbf{n}\|_{0,\Sigma} \,\leq\, \frac{2\mu\alpha_{\mathrm{f}}}{C_{\mathrm{tr}}^2\, C_{\mathrm{Sob},\Sigma}^2}.$$

There holds

$$A_{\mathrm{F}}(\mathbf{v},\mathbf{v}) \,+\, O_{\mathrm{F}}^h(\mathbf{w}_{\mathrm{f}};\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) \,\geq\, \alpha_{\mathrm{F}}\,\|\mathbf{v}\|_{\mathbf{H}}^2 \quad \forall\, \mathbf{v} \in \mathbf{V}_h.$$

Now we establish some useful properties of O_{T}^h, similar to the ones provided in Lemma 3.2.2.

Lemma 3.3.2. *If* $\mathbf{w} \in \mathbf{V}_h$ *is given, then the following identities hold*

$$O_{\mathrm{T}}^h(\mathbf{w};\theta,\psi) \,=\, -\,O_{\mathrm{T}}^h(\mathbf{w};\psi,\theta), \tag{3.71a}$$

$$|O_{\mathrm{T}}^h(\mathbf{w};\theta,\psi)| \,\leq\, \widehat{C}_{O_{\mathrm{T}}^1}\|\mathbf{w}\|_{\mathbf{H}}\|\psi\|_{1,\Omega}\left(\|\theta\|_{0,3,\Omega_{\mathrm{f}}} + \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}\right), \tag{3.71b}$$

$$|O_{\mathrm{T}}^h(\mathbf{w};\theta,\psi)| \,\leq\, \widehat{C}_{O_{\mathrm{T}}^2}\,\|\mathbf{w}\|_{\mathbf{H}}\,\|\psi\|_{1,\Omega}\,\left(\|\theta\|_{1,\Omega_{\mathrm{f}}} \,+\, \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}\right), \tag{3.71c}$$

for all θ, $\psi \in \Psi_{h,0}$ *where* $\widehat{C}_{O_{\mathrm{T}}^1} := C_{\mathrm{Sob},\Omega_{\mathrm{f}}}\left(1 + \frac{\sqrt{d}}{2}\right)$ *and* $\widehat{C}_{O_{\mathrm{T}}^2} := C_{\mathrm{Sob},\Omega_{\mathrm{f}}}^2\left(1 + \frac{\sqrt{d}}{2}\right)$.

Proof. Given $\mathbf{w} \in \mathbf{V}_h$, we recall that $\operatorname{div}\mathbf{w}_{\mathrm{m}} = 0$ in Ω_{m}. Then, integrating by parts one can easily obtain (3.71a). In turn, using (3.71a) and proceeding analogously to the proof of estimate (3.13a), it is easy to deduce (3.71b). Finally, by combining (3.71b) and (1.7), we obtain (3.71c). □

Observe that, similarly to the continuous case, by combining (3.71a) and the third estimate in (3.70), for a given $\mathbf{w} \in \mathbf{V}_h$, it is possible to obtain

$$A_{\mathrm{T}}(\psi,\psi) \,+\, O_{\mathrm{T}}^h(\mathbf{w};\psi,\psi) \,\geq\, \alpha_{\mathrm{T}}\,\|\psi\|_{1,\Omega}^2 \qquad \forall\, \psi \in \Psi_{h,0}. \tag{3.72}$$

We conclude this section by establishing the discrete counterpart of (3.11).

Lemma 3.3.3. *Given* $h = \max\{h_{\mathrm{f}}, h_{\mathrm{m}}\} > 0$*, let* $\mathbf{H}_h$ *and* Q_h *be the discrete space introduced in Section 3.3.1 and, in particular for the three-dimensional case, assume that the meshes* Σ_h *and* $\Sigma_{\widehat{h}}$ *satisfy* (3.64) *and that* Σ_h *is quasi-uniform. Then, there exist* $h_0 > 0$ *and* $\widehat{\beta} > 0$*, independent of* h*, such that for any* $h_{\mathrm{m}} \leq h_0$*, there holds:*

$$\sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \,\geq\, \widehat{\beta}\,\|(q,\xi)\|_{\mathrm{Q}} \quad \forall\,(q,\xi) \in \mathrm{Q}_h. \tag{3.73}$$

Proof. We begin by observing that the proof of (3.73) for the $2D$ case, can be found in [45, Lemma 9], which is based on the proof of [56, Lemma 4.3]. Now, for the $3D$ case, we let $(q,\xi) \in \mathrm{Q}_h$ and recall from [90, Theorem 4.1.9] that there exists a discrete lifting $\mathbf{L}_h : \Phi_h(\Sigma) \to \mathbf{H}_h(\Omega_\mathrm{m})$, such that for all $\phi_h \in \Phi_h(\Sigma)$, there hold

$$\|\mathbf{L}_h(\phi_h)\|_{\mathrm{div};\Omega_\mathrm{m}} \le c_\star \|\phi_h\|_{-1/2,\Sigma} \quad \text{and} \quad \mathbf{L}_h(\phi_h)\cdot\mathbf{n} = \phi_h \quad \text{on} \quad \Sigma. \tag{3.74}$$

Then, for a given $\phi_h \in \Phi_h(\Sigma)$, we proceed analogously to the proof of [45, Lemma 7] and let $\widehat{\mathbf{v}} := (\mathbf{0}, \mathbf{L}_h(\phi_h))$, to deduce that

$$\begin{aligned}\sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_\mathbf{H}} &\ge \frac{|B(\widehat{\mathbf{v}},(q,\xi))|}{\|\widehat{\mathbf{v}}\|_\mathbf{H}} \\ &= \frac{|\langle \mathbf{L}_h(\phi_h)\cdot\mathbf{n},\xi\rangle_\Sigma + (\mathrm{div}(\mathbf{L}_h(\phi_h)),q)_{\Omega_\mathrm{m}}|}{\|\mathbf{L}_h(\phi_h)\|_{\mathrm{div};\Omega_\mathrm{m}}} \\ &\ge \frac{1}{c_\star}\frac{|\langle\phi_h,\xi\rangle_\Sigma|}{\|\phi_h\|_{-1/2,\Sigma}} - \|q\|_{0,\Omega}.\end{aligned} \tag{3.75}$$

Then, noticing that ϕ_h is arbitrary, from (3.63) we obtain

$$\sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_\mathbf{H}} \ge c_1\|\xi\|_{1/2,\Sigma} - \|q\|_{0,\Omega}, \tag{3.76}$$

with $c_1 > 0$, independent of h.

Now, let us recall that the Bernardi-Raugel and Raviart-Thomas interpolation operators, denoted respectively by $\Pi_\mathrm{f} : \mathbf{H}^1(\Omega_\mathrm{f}) \to \mathbf{H}_h(\Omega_\mathrm{f})$ and $\Pi_\mathrm{m} : \mathbf{H}^1(\Omega_\mathrm{m}) \to \mathbf{H}_h(\Omega_\mathrm{m})$ (see [11] and [62]), satisfy

$$\int_F \Pi_\star(\mathbf{v}_\star)\cdot\mathbf{n} = \int_F \mathbf{v}_\star\cdot\mathbf{n} \qquad \text{for each face } F \text{ of } \mathcal{T}_h^\star \quad \text{and} \quad \forall\,\mathbf{v}_\star \in \mathbf{H}^1(\Omega_\star)\,, \tag{3.77}$$

and

$$\int_{\Omega_\star} q\,\mathrm{div}(\Pi_\star(\mathbf{v}_\star)) = \int_{\Omega_\star} q\,\mathrm{div}\,\mathbf{v}_\star \qquad \forall\, q \in \mathrm{L}_h(\Omega) \quad \text{and} \quad \forall\,\mathbf{v}_\star \in \mathbf{H}^1(\Omega_\star)\,, \tag{3.78}$$

with $\star \in \{\mathrm{f},\mathrm{m}\}$ and that Π_m satisfies the approximation property (see [52, Lemma 3.19] and [69, Theorem 3.16]):

$$\|\mathbf{v}_\mathrm{m} - \Pi_\mathrm{m}(\mathbf{v}_\mathrm{m})\|_{0,\Omega_\mathrm{m}} \le C\,h_\mathrm{m}^\varepsilon\,\{|\mathbf{v}_\mathrm{m}|_{\varepsilon,\Omega_\mathrm{m}} + \|\mathrm{div}\,\mathbf{v}_\mathrm{m}\|_{0,\Omega_\mathrm{m}}\}\,, \tag{3.79}$$

for all $\mathbf{v}_\mathrm{m} \in \mathbf{H}^\varepsilon(\Omega_\mathrm{m}) \cap \mathbf{H}(\mathrm{div};\Omega_\mathrm{m})$, $\forall\,\varepsilon \in (0,1)$. We also recall that using a duality argument, it can be proved that the L^2-orthogonal projection $\mathcal{P}_\Sigma : \mathrm{L}^2(\Sigma) \to \Phi_h(\Sigma)$ satisfies

$$\|z - \mathcal{P}_\Sigma(z)\|_{-1/2,\Sigma} \le C h_\mathrm{m}^{1/2}\|z\|_{0,\Sigma} \quad \forall\, z \in \mathrm{L}^2(\Sigma). \tag{3.80}$$

Then, proceeding analogously to the proof of [45, Lemma 8] and [56, Lemma 4.2], we let $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ be such that

$$\operatorname{div} \mathbf{z} = -q \quad \text{in} \quad \Omega \quad \text{and} \quad \|\mathbf{z}\|_{1,\Omega} \leq C\|q\|_{0,\Omega}, \tag{3.81}$$

let $\varphi \in \mathrm{H}^{1+\varepsilon}(\Omega_{\mathrm{m}})$, with $\varepsilon \in [0, \frac{\pi}{\omega})$, be the unique solution of the variational problem

$$-\Delta\varphi = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}, \qquad \frac{\partial \varphi}{\partial \mathbf{n}} = \begin{cases} (\Pi_{\mathrm{f}}(\mathbf{z}_{\mathrm{f}}) - \Pi_{\mathrm{m}}(\mathbf{z}_{\mathrm{m}})) \cdot \mathbf{n} & \text{on} \quad \Sigma \\ 0 \quad \text{on} \quad \Gamma_{\mathrm{m}} \end{cases}, \qquad \int_{\Omega_{\mathrm{m}}} \varphi = 0,$$

where ω is the largest interior angle of Ω_{m}, and define

$$\mathbf{w}_{\mathrm{f}} := \Pi_{\mathrm{f}}(\mathbf{z}_{\mathrm{f}}), \qquad \mathbf{w}_{\mathrm{m}} := \Pi_{\mathrm{m}}(\mathbf{z}_{\mathrm{m}}) + \Pi_{\mathrm{m}}(\nabla\varphi),$$

where $\mathbf{z}_\star = \mathbf{z}|_{\Omega\star}$, for $\star \in \{\mathrm{f}, \mathrm{m}\}$. Then, we apply (3.77)–(3.81), the boundedness of Π_{f}, and the trace inequality, to obtain

$$(\operatorname{div} \mathbf{w}_{\mathrm{f}}, q)_{\Omega_{\mathrm{f}}} + (\operatorname{div} \mathbf{w}_{\mathrm{m}}, q)_{\Omega_{\mathrm{m}}} = -\|q\|_{0,\Omega}^2, \qquad \|(\mathbf{w}_{\mathrm{f}}, \mathbf{w}_{\mathrm{m}})\|_{\mathbf{H}} \leq C\|q\|_{0,\Omega}, \tag{3.82}$$

and

$$\begin{aligned} |\langle \mathbf{w}_{\mathrm{f}} \cdot \mathbf{n} - \mathbf{w}_{\mathrm{m}} \cdot \mathbf{n}, \xi \rangle_\Sigma| &\leq \|\Pi_{\mathrm{f}}(\mathbf{z}_{\mathrm{f}}) \cdot \mathbf{n} - \mathcal{P}_\Sigma(\Pi_{\mathrm{f}}(\mathbf{z}_{\mathrm{f}}) \cdot \mathbf{n})\|_{-1/2,\Sigma} \|\xi\|_{1/2,\Sigma} \\ &\leq c h_{\mathrm{m}}^{1/2} \|\Pi_{\mathrm{f}}(\mathbf{z}_{\mathrm{f}})\|_{1,\Omega} \|\xi\|_{1/2,\Sigma} \\ &\leq C h_{\mathrm{m}}^{1/2} \|q\|_{0,\Omega} \|\xi\|_{1/2,\Sigma}. \end{aligned} \tag{3.83}$$

Thus, by applying (3.82) and (3.83) we deduce that

$$\sup_{\mathbf{v} \in \mathbf{H}_h \setminus \{\mathbf{0}\}} \frac{B(\mathbf{v}, (q, \xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \frac{|B((\mathbf{w}_{\mathrm{f}}, \mathbf{w}_{\mathrm{m}}), (q, \xi))|}{\|(\mathbf{w}_{\mathrm{f}}, \mathbf{w}_{\mathrm{m}})\|_{\mathbf{H}}} \geq c_2\|q\|_{0,\Omega} - c_3 h_m^{1/2} \|\xi\|_{1/2,\Sigma}, \tag{3.84}$$

with $c_2, c_3 > 0$ independent of h.

In this way, combining (3.76) and (3.84) and proceeding analogously to the proof of [45, Lemma 9], we obtain that for any $h_{\mathrm{m}} \leq h_0$, with $h_0 = \left(\frac{c_1 c_2}{2c_3}\right)^2$, the inf-sup condition (3.73) holds. □

The discrete lifting

For the sake of the subsequent analysis, and analogously to the continuous case, given $\delta > 0$ now we introduce the discrete extension operator $\mathrm{E}_{\delta,h} : \mathrm{W}^{3/4,4}(\Gamma) \to \Psi_h$ given by $\mathrm{E}_{\delta,h} := \mathrm{I}_h \mathrm{E}_\delta$, where E_δ is the extension operator defined in (3.21) and

I_h is the Lagrange interpolation operator. Then, it is clear from (3.65) that there holds

$$\theta^{\delta}_{\mathrm{D},h} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})|_{\Gamma}. \tag{3.85}$$

In what follows we derive some useful estimates for the operator $\mathrm{E}_{\delta,h}$ that will allow us to prove existence and stability of solution of problem (3.66). To that end we first recall that the Lagrange operator I_h satisfies the following approximation property (see, eg. [47, Theorem 1.103]): Given $p > d$, $l \in \{0,1\}$, $0 \leq m \leq l+1$ and $T \in \mathcal{T}_h$, there holds

$$|\mathrm{I}_h(\psi) - \psi|_{m,p,T} \leq c h_T^{l+1-m} |\psi|_{l+1,p,T} \quad \forall\, \psi \in \mathrm{W}^{l+1,p}(T). \tag{3.86}$$

In addition, we recall the following inverse inequality (see [47, Lemma 1.138]):

$$\|\psi_h\|_{l,p,T} \leq C h_T^{m-l+d(\frac{1}{p}-\frac{1}{q})} \|\psi_h\|_{m,q,T} \qquad \forall\, \psi_h \in \Psi_h, \tag{3.87}$$

for any $T \in \mathcal{T}_h$, $l \geq 0$, $0 \leq m \leq l$ and $p, q \geq 1$.

Finally, for all $\psi \in \mathrm{W}^{1,4}(\Omega) \subseteq \mathrm{C}^0(\overline{\Omega})$, it is easy to see that

$$\|\mathrm{I}_h(\psi)\|_{0,\infty,T} \leq (d+1)\|\psi\|_{0,\infty,T} \quad \forall\, T \in \mathcal{T}_h. \tag{3.88}$$

Lemma 3.3.4. *The following estimates hold:*

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,3,\Omega} \;\leq\; \widehat{C}_{\mathrm{lift},1}\, \delta^{1/12}(h\delta^{-1} + h + 1)\, \|\zeta\|_{3/4,4,\Gamma}, \tag{3.89a}$$

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{1,\Omega} \;\leq\; \widehat{C}_{\mathrm{lift},2}\, \delta^{1/4}(2 + \delta^{-1})\, \|\zeta\|_{3/4,4,\Gamma}, \tag{3.89b}$$

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,\infty,\Omega} \;\leq\; (d+1)\|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta}, \tag{3.89c}$$

for all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$, where $\widehat{C}_{\mathrm{lift},1}$, $\widehat{C}_{\mathrm{lift},2} > 0$ are constants indepedent of h and δ.

Proof. Given $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$, we first use estimate (3.86) and recall that $\mathrm{E}_\delta = \beta_\delta \mathrm{E}$ (cf. (3.21)), to obtain

$$\begin{aligned}
\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,3,\Omega} &= \|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|_{0,3,\Omega} \leq \|\mathrm{I}_h(\mathrm{E}_\delta(\zeta)) - \mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} + \|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} \\
&\leq ch|\mathrm{E}_\delta(\zeta)|_{1,3,\Omega} + \|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} \\
&\leq ch(\|(\nabla\beta_\delta)\mathrm{E}(\zeta)\|_{0,3,\Omega_\delta} + \|\beta_\delta \nabla \mathrm{E}(\zeta)\|_{0,3,\Omega_\delta}) + \|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega}.
\end{aligned} \tag{3.90}$$

Now, for the first term in the above inequality Hölder's inequality we have

$$\begin{aligned}
\|(\nabla\beta_\delta)\mathrm{E}(\zeta)\|^3_{0,3,\Omega_\delta} &= \int_{\Omega_\delta} |\nabla\beta_\delta|^3 |\mathrm{E}(\zeta)|^3 \\
&\leq \||\nabla\beta_\delta|^3\|_{0,4,\Omega_\delta} \||\mathrm{E}(\zeta)|^3\|_{0,4/3,\Omega} \\
&= \|\nabla\beta_\delta\|^3_{0,12,\Omega_\delta} \|\mathrm{E}(\zeta)\|^3_{0,4,\Omega}.
\end{aligned}$$

Then, we recall that β_δ (cf. (3.19)) satisfies $|\nabla\beta_\delta| = \delta^{-1}$ a.e in $\{\mathbf{x} \in \Omega_\delta : \delta \leq \mathrm{dist}(\mathbf{x},\Gamma) \leq 2\delta\}$ and $\nabla\beta_\delta$ vanishes elsewhere, to obtain

$$\|(\nabla\beta_\delta)\mathrm{E}(\zeta)\|_{0,3,\Omega_\delta} \leq \delta^{-1}|\Omega_\delta|^{1/12}\|\mathrm{E}(\zeta)\|_{0,4,\Omega}.$$

Similarly, using again Hölder's inequality and the fact that $\beta_\delta \leq 1$, we have

$$\|\beta_\delta\nabla\mathrm{E}(\zeta)\|^3_{0,3,\Omega_\delta} \leq \||\beta_\delta|^3\|_{0,4,\Omega_\delta}\||\nabla\mathrm{E}(\zeta)|^3\|_{0,4/3,\Omega} \leq |\Omega_\delta|^{1/4}\|\nabla\mathrm{E}(\zeta)\|^3_{0,4,\Omega},$$

which implies

$$\|\beta_\delta\nabla\mathrm{E}(\zeta)\|_{0,3,\Omega_\delta} \leq |\Omega_\delta|^{1/12}|\mathrm{E}(\zeta)|_{1,4,\Omega}. \tag{3.91}$$

In this way, combining (3.90)–(3.91), applying (3.23a) and employing (3.18) and the fact that $|\Omega_\delta| \approx \delta$, we readily obtain (3.89a).

To derive (3.89b) we first recall that $\mathcal{T}_h^\delta \subseteq \mathcal{T}_h$, make use the fact that $\mathrm{E}_\delta(\zeta) = 0$ in $\Omega\backslash\Omega_\delta$ and employ estimate (3.87), to obtain

$$\begin{aligned} |\mathrm{E}_{\delta,h}(\zeta)|^2_{1,\Omega} &= |\mathrm{I}_h(\mathrm{E}_\delta(\zeta))|^2_{1,\Omega_\delta} = \sum_{T\in\mathcal{T}_h^\delta} |\mathrm{I}_h(\mathrm{E}_\delta(\zeta))|^2_{1,T} \\ &\leq \sum_{T\in\mathcal{T}_h^\delta} h_T^{d/2}\|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|^2_{1,4,T} \leq \left(\sum_{T\in\mathcal{T}_h^\delta} h_T^d\right)^{1/2} \|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|^2_{1,4,\Omega_\delta}, \end{aligned}$$

which combined with the fact that $h_T^d \approx |T|$ and estimate (3.86) with $p = 4$, implies

$$\begin{aligned} |\mathrm{E}_{\delta,h}(\zeta)|_{1,\Omega} \leq\ & C|\Omega_\delta|^{1/4}\|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|_{1,4,\Omega_\delta} \\ \leq\ & C\delta^{1/4}(\|\mathrm{I}_h(\mathrm{E}_\delta(\zeta)) - \mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta} + \|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta}) \\ \leq\ & C_1\delta^{1/4}|\mathrm{E}_\delta(\zeta)|_{1,4,\Omega_\delta} + C_2\delta^{1/4}\|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta} \\ \leq\ & \delta^{1/4}\hat{C}\|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta}. \end{aligned} \tag{3.92}$$

Then, using again that $\beta_\delta \leq 1$ in Ω_δ, $|\nabla\beta_\delta| = \delta^{-1}$ a.e in $\{\mathbf{x} \in \Omega_\delta : \delta \leq \mathrm{dist}(\mathbf{x},\Gamma) \leq 2\delta\}$ and $\nabla\beta_\delta$ vanishes elsewhere, there holds

$$\begin{aligned} \|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta} &= \|\beta_\delta\mathrm{E}(\zeta)\|_{1,4,\Omega_\delta} \\ &\leq \|\beta_\delta\mathrm{E}(\zeta)\|_{0,4,\Omega_\delta} + \|\beta_\delta\nabla\mathrm{E}(\zeta)\|_{0,4,\Omega_\delta} + \|(\nabla\beta_\delta)\mathrm{E}(\zeta)\|_{0,4,\Omega_\delta} \\ &\leq (2+\delta^{-1})\|\mathrm{E}(\zeta)\|_{1,4,\Omega}, \end{aligned}$$

which together with (3.92) and (3.18), imply (3.89b).

Finally, for (3.89c) we first notice that, since $\mathcal{T}_h^\delta \subseteq \mathcal{T}_h$ and $\mathrm{E}_\delta(\zeta) = 0$ in $\Omega\backslash\Omega_\delta$, then

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,\infty,\Omega} = \|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|_{0,\infty,\Omega_\delta} = \|\mathrm{I}_h(\beta_\delta \mathrm{E}(\zeta))\|_{0,\infty,\Omega_\delta}.$$

Then, employing the second property of β_δ in (3.20) and using the fact that $\beta_\delta \mathrm{E}(\zeta)$ is continuous, from the identity above we readily obtain (3.89c).

□

Main result

Similarly to the continuous case, let us fix $\delta > 0$ (to be specified below in Theorem 3.3.6) and decompose the discrete temperature θ_h as $\theta_h = \theta_{h,0} + \theta_{h,1}$, with $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D},h}) \in \Psi_h$ and $\theta_{h,0} = \theta_h - \theta_{h,1} \in \Psi_{h,0}$ and analogously to the analysis of the continuous problem we introduce the reduced version of problem (3.66) which reads:

Find $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ such that

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u}_h, \mathbf{v}) \,+\, O_{\mathrm{F}}^h(\mathbf{u}_{h,\mathrm{f}}; \mathbf{u}_{h,\mathrm{f}}, \mathbf{v}_\mathrm{f}) \,-\, D(\theta_{h,0}, \mathbf{v}) \;&=\; D(\theta_{h,1}, \mathbf{v}), \\ A_{\mathrm{T}}(\theta_{h,0}, \psi) \,+\, O_{\mathrm{T}}^h(\mathbf{u}_h; \theta_{h,0} + \theta_{h,1}, \psi) \;&=\; -\, A_{\mathrm{T}}(\theta_{h,1}, \psi), \end{aligned} \tag{3.93}$$

for all $\mathbf{v} \in \mathbf{V}_h$ and forall $\psi \in \Psi_{h,0}$, where $\mathbf{V}_h$ is the discrete kernel of B defined in (3.69).

Using the discrete inf-sup condition (3.73) and analogously to the continuous case we readily obtain that both problems (3.66) and (3.93) are equivalent. This result is established next.

Lemma 3.3.5. *Given $h = \max\{h_\mathrm{f}, h_\mathrm{m}\} > 0$, let $\mathbf{H}_h$ and Q_h be the discrete space introduced in Section 3.3.1 and, in particular for the three-dimensional case, assume that the meshes Σ_h and $\Sigma_{\widehat{h}}$ satisfy* (3.64) *and that Σ_h is quasi-uniform. In addition, let $h_0 > 0$ be the fixed constant provided by Lemma 3.3.3, and assume that $h_\mathrm{m} \leq h_0$, so that inf-sup condition* (3.73) *holds true. If $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ is a solution of* (3.66)*, then $\mathbf{u}_h \in \mathbf{V}_h$ and $(\mathbf{u}_h, \theta_{h,0}) = (\mathbf{u}_h, \theta_h - \theta_{h,1})$ is a solution to* (3.93)*. Conversely, if $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ is a solution of* (3.93)*, then there exists $(p_h, \lambda_h) \in \mathrm{Q}_h$, such that $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) = (\mathbf{u}_h, (p_h, \lambda_h), \theta_{h,0} + \theta_{h,1})$ is a solution to* (3.66)*.*

Notice that since (3.93) is a finite-dimensional problem, we can use the same strategy employed in Section 3.2.2 to analyze problem (3.27). To that end, let us now define the compact and convex set

$$\mathbf{X}_h := \left\{ (\mathbf{w}, \phi) \in \mathbf{V}_h \times \Psi_{h,0} : \quad \begin{array}{l} \|\mathbf{w}\|_{\mathbf{H}} \leq C_{\mathbf{u}}(\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}})\|\theta_{h,1}\|_{1,\Omega}, \\ \text{and } \; \|\phi\|_{1,\Omega} \,\leq\, C_\theta \, \|\theta_{h,1}\|_{1,\Omega} \end{array} \right\}, \tag{3.94}$$

with $C_{\mathbf{u}}$ and C_θ defined in (3.28) and (3.29), respectively, and the operator $\mathscr{J}_h$: $\mathbf{X}_h \to \mathbf{V}_h \times \Psi_{h,0}$ given by

$$\mathscr{J}_h(\mathbf{w}, \phi) = (\mathbf{u}_h, \theta_{h,0}) \qquad \forall\, (\mathbf{w}, \phi) \in \mathbf{X}_h,$$

where $(\mathbf{u}_h, \theta_{h,0})$ is the solution of the linearized version of problem (3.93):

Find $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ such that

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u}_h, \mathbf{v}) + O^h_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}}; \mathbf{u}_{h,\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) &= D(\phi, \mathbf{v}) + D(\theta_{h,1}, \mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{V}_h, \\ A_{\mathrm{T}}(\theta_{h,0}, \psi) + O^h_{\mathrm{T}}(\mathbf{w}; \theta_{h,0}, \psi) &= -A_{\mathrm{T}}(\theta_{h,1}, \psi) - O^h_{\mathrm{T}}(\mathbf{w}; \theta_{h,1}, \psi) && \forall\, \psi \in \Psi_{h,0}. \end{aligned}$$

Now we state the main result of this section.

Theorem 3.3.6. *Given $h = \max\{h_{\mathrm{f}}, h_{\mathrm{m}}\} > 0$, let $\mathbf{H}_h$ and Q_h be the discrete space introduced in Section 3.3.1 and, in particular for the three-dimensional case, assume that the meshes Σ_h and $\Sigma_{\widehat{h}}$ satisfy* (3.64) *and that Σ_h is quasi-uniform. In addition, let $h_0 > 0$ be the fixed constant provided by Lemma 3.3.3, and assume that $h \le \min\{h_0, \delta\}$, with $\delta > 0$ satisfying*

$$\frac{C_D \widehat{C}_{O^1_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\alpha_{\mathrm{T}}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\widehat{C}_{\mathrm{lift},1}\, \delta^{1/12}(2+\delta)\, \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \le \frac{1}{4}. \tag{3.95}$$

Let $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ be such that

$$\frac{C_D\, \widehat{C}_{O^1_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\, \alpha_{\mathrm{T}}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_{h,1}\|_{0,\infty,\Omega} \le \frac{1}{4}, \tag{3.96}$$

and assume further that $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ satisfies

$$C_{\mathbf{u}}\,(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\, \widehat{C}_{\mathrm{lift},2}\, \delta^{1/4}(2+\delta^{-1})\, \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \le \frac{2\mu\alpha_{\mathrm{f}}}{C^3_{\mathrm{tr}}\, C^2_{\mathrm{Sob},\Sigma}}. \tag{3.97}$$

Then, there exists at least one $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ solution to (3.66). *Moreover, there exists $C > 0$, independent of h, such that*

$$\|\mathbf{u}_h\|_{\mathbf{H}} + \|(p_h, \lambda_h)\|_{\mathrm{Q}} + \|\theta_h\|_{1,\Omega} \le C\, \left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}} + \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}\right). \tag{3.98}$$

Proof. First, let us observe that assumption $h \le \min\{h_0, \delta\}$ implies that estimate (3.89a), with $\zeta = \theta_{\mathrm{D}}$, becomes

$$\|\theta_{h,1}\|_{0,3,\Omega} = \|\mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})\|_{0,3,\Omega} \le \widehat{C}_{\mathrm{lift},1}\, \delta^{1/12}(2+\delta)\, \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}.$$

Then, for a given $(\mathbf{w}, \phi) = ((\mathbf{w}_{\mathrm{f}}, \mathbf{w}_{\mathrm{m}}), \phi) \in \mathbf{X}_h$, analogously to the proof of Lemma 3.2.6 we make use of assumptions (3.95), (3.96), (3.97) and the aforementioned inequality to deduce that there exists a unique $(\mathbf{u},\theta)\in\mathbf{X}_h$, such that $\mathcal{J}_h(\mathbf{w},\phi)=(\mathbf{u},\theta)$, thus $\mathcal{J}_h$ is well-defined and satisfies $\mathcal{J}_h(\mathbf{X}_h)\subseteq\mathbf{X}_h$. In addition, using the same arguments employed in the proof of Lemma 3.2.7, we can deduce that $\mathcal{J}_h$ is continuous. According to the above, and analogously to the proof of Theorem 3.2.8, we employ Theorem 3.2.5 to obtain that there exists $(\mathbf{u}_h,\theta_{h,0})\in\mathbf{X}_h$ such that (3.93) holds, which together with Lemma 3.3.5 implies that there exists $(p_h,\lambda_h)\in\mathrm{Q}_h$, such that $(\mathbf{u}_h,(p_h,\lambda_h),\theta_h)=(\mathbf{u}_h,(p_h,\lambda_h),\theta_{h,0}+\theta_{h,1})$ is a solution to (3.66).

Finally, using the fact that $(\mathbf{u}_h,\theta_{h,0})\in\mathbf{X}_h$ and proceeding analogously to the proof of Corollary 3.2.10 we easily deduce that $(\mathbf{u}_h,(p_h,\lambda_h),\theta_h)$ satisfies (3.98), which concludes the proof. □

Remark 3.3.2. *Observe that, owing to* (3.89c) *and similarly to Remark 3.2.1, assumption* (3.96) *becomes*

$$\frac{(d+1)C_D\,\widehat{C}_{O^1_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\,\alpha_{\mathrm{T}}}\left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}}+\|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}\right)\|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta}\leq\frac{1}{4},$$

and again one could take an small enough δ in such a way $\|\mathrm{E}(\theta_{\mathrm{D}})\|_{0,\infty,\Omega_\delta}\approx\|\theta_{\mathrm{D}}\|_{0,\infty,\Gamma}$, and simply assume that

$$\frac{(d+1)C_D\,\widehat{C}_{O^1_{T}}}{\alpha_{\mathrm{F}}\,\alpha_{\mathrm{T}}}\left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}}+\|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}\right)\|\theta_{\mathrm{D}}\|_{0,\infty,\Gamma}\leq\frac{1}{4}.$$

3.4 Error analysis

In this section we address the error analysis and provide the theoretical rate of convergence for the Galerkin scheme (3.66). We begin with some notations and preliminary results.

Let us assume that the hypotheses of Theorem 3.2.11 hold and let $(\mathbf{u},(p,\lambda),\theta)=(\mathbf{u},(p,\lambda),\theta_0+\theta_1)\in\mathbf{H}\times\mathrm{Q}\times\mathrm{H}^1(\Omega)$ be a solution to (3.2), with $\theta_1=\mathrm{E}_\delta(\theta_{\mathrm{D}})\in\mathrm{W}^{1,4}(\Omega)$ and $\theta_0\in\mathrm{H}^1_0(\Omega)$. In addition, we assume that the hypotheses of Theorem 3.3.6 and let $(\mathbf{u}_h,(p_h,\lambda_h),\theta_h)=(\mathbf{u}_h,(p_h,\lambda_h),\theta_{h,0}+\theta_{h,1})\in\mathbf{H}_h\times\mathrm{Q}_h\times\Psi_h$ be a solution to (3.66), with $\theta_{h,1}=\mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})\in\Psi_h$ and $\theta_{h,0}\in\Psi_{h,0}$. Then to simplify the subsequent analysis, we write

$$\begin{gathered}\mathbf{e}_{\mathbf{u}_{\mathrm{f}}}=\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}},\qquad\mathbf{e}_{\mathbf{u}_{\mathrm{m}}}=\mathbf{u}_{\mathrm{m}}-\mathbf{u}_{h,\mathrm{m}},\qquad\mathrm{e}_p=p-p_h,\\ \mathrm{e}_\lambda=\lambda-\lambda_h,\qquad\mathrm{e}_\theta=\theta-\theta_h.\end{gathered}\tag{3.99}$$

We decompose these errors as

$$\begin{gathered}\mathbf{e}_{\mathbf{u}_{\mathrm{f}}}=\boldsymbol{\varrho}_{\mathbf{u}_{\mathrm{f}}}+\boldsymbol{\chi}_{\mathbf{u}_{\mathrm{f}}},\qquad\mathbf{e}_{\mathbf{u}_{\mathrm{m}}}=\boldsymbol{\varrho}_{\mathbf{u}_{\mathrm{m}}}+\boldsymbol{\chi}_{\mathbf{u}_{\mathrm{m}}},\qquad\mathrm{e}_p=\varrho_p+\chi_p,\\ \mathrm{e}_\lambda=\varrho_\lambda+\chi_\lambda,\qquad\mathrm{e}_\theta=\varrho_\theta+\chi_\theta,\end{gathered}\tag{3.100}$$

with

$$
\begin{aligned}
\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{f}} &= \mathbf{u}_\mathrm{f} - \widehat{\mathbf{v}}_{h,\mathrm{f}}, & \boldsymbol{\chi}_{\mathbf{u}_\mathrm{f}} &= \widehat{\mathbf{v}}_{h,\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}, \\
\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{m}} &= \mathbf{u}_\mathrm{m} - \widehat{\mathbf{v}}_{h,\mathrm{m}}, & \boldsymbol{\chi}_{\mathbf{u}_\mathrm{m}} &= \widehat{\mathbf{v}}_{h,\mathrm{m}} - \mathbf{u}_{h,\mathrm{m}}, \\
\varrho_p &= p - \widehat{q}_h, & \chi_p &= \widehat{q}_h - p_h, \\
\varrho_\lambda &= \lambda - \widehat{\xi}_h, & \chi_\lambda &= \widehat{\xi}_h - \lambda_h, \\
\varrho_\theta &= \theta - \widehat{\psi}_h, & \chi_\theta &= \widehat{\psi}_h - \theta_h,
\end{aligned}
\tag{3.101}
$$

where $\widehat{\mathbf{v}}_h = (\widehat{\mathbf{v}}_{h,\mathrm{f}}, \widehat{\mathbf{v}}_{h,\mathrm{m}}) \in \mathbf{V}_h$, $(\widehat{q}_h, \widehat{\xi}_h) \in \mathrm{Q}_h$, and $\widehat{\psi}_h \in \Psi_h^\Gamma$. Here, Ψ_h^Γ is the set of functions in Ψ_h that coincide with $\theta_{\mathrm{D},h}^\delta$ (cf. (3.85)) on Γ, that is

$$
\Psi_h^\Gamma := \{\psi_h \in \Psi_h \,:\, \psi_h|_\Gamma = \theta_{\mathrm{D},h}^\delta\}.
$$

Finally, we let

$$
\mathbf{e}_\mathbf{u} = (\mathbf{e}_{\mathbf{u}_\mathrm{f}}, \mathbf{e}_{\mathbf{u}_\mathrm{m}}), \quad \boldsymbol{\varrho}_\mathbf{u} = (\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{f}}, \boldsymbol{\varrho}_{\mathbf{u}_\mathrm{m}}) \quad \text{and} \quad \boldsymbol{\chi}_\mathbf{u} = (\boldsymbol{\chi}_{\mathbf{u}_\mathrm{f}}, \boldsymbol{\chi}_{\mathbf{u}_\mathrm{m}}).
$$

Let us recall that, since the inf-sup condition (3.73) holds, it is possible to prove that there exists $c > 0$, independent of h, such that (see for instance [52, Theorem 2.6])

$$
\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_\mathbf{H} \le c \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_\mathbf{H}. \tag{3.102}
$$

Now we provide some useful properties of the convective terms that will allow us to derive the desired error estimates.

Lemma 3.4.1. *Let $\mathbf{u} = (\mathbf{u}_\mathrm{f}, \mathbf{u}_\mathrm{m}) \in \mathbf{V}$ and $\theta \in \mathrm{H}^1(\Omega)$. The following identities hold*

$$
\begin{aligned}
O_\mathrm{F}^h(\mathbf{u}_\mathrm{f}; \mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f}) &= O_\mathrm{F}(\mathbf{u}_\mathrm{f}; \mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f}) & \forall\, \mathbf{v}_\mathrm{f} \in \mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f}), \\
O_\mathrm{T}^h(\mathbf{u}; \theta, \psi) &= O_\mathrm{T}(\mathbf{u}; \theta, \psi) & \forall\, \psi \in \Psi_h.
\end{aligned}
\tag{3.103}
$$

In addition, for any $\mathbf{u}_h \in \mathbf{H}_h$ and $\theta \in \mathrm{H}^1(\Omega)$, such that $\theta|_{\Omega_\mathrm{m}} \in \mathrm{W}^{1,4}(\Omega_\mathrm{m})$, there holds

$$
|O_\mathrm{T}^h(\mathbf{u}_h; \theta, \psi)| \le \widehat{C}_{O_\mathrm{T}^3} \|\mathbf{u}_h\|_\mathbf{H} \, (\|\theta\|_{1,\Omega_\mathrm{f}} + \|\theta\|_{1,4,\Omega_\mathrm{m}}) \, \|\psi\|_{1,\Omega} \quad \forall\, \psi \in \Psi_{h,0}, \tag{3.104}
$$

where $\widehat{C}_{O_\mathrm{T}^3}$ is a positive constant independent of h.

Proof. Given $\mathbf{u} = (\mathbf{u}_\mathrm{f}, \mathbf{u}_\mathrm{m}) \in \mathbf{V}$ and $\theta \in \mathrm{H}^1(\Omega)$, it is clear that $\mathrm{div}\, \mathbf{u}_\star = 0$ in $\Omega_\star$ for $\star \in \{\mathrm{f}, \mathrm{m}\}$, which readily implies (3.103). On the other hand, given $\mathbf{u}_h \in \mathbf{H}_h$ and employing the Hölder inequality (1.1), we have that for all $\psi \in \Psi_{h,0}$, there holds

$$\begin{aligned} |O_{\mathrm{T}}^h(\mathbf{u}_h;\theta,\psi)| &\leq |(\mathbf{u}_{h,\mathrm{f}}\cdot\nabla\theta,\psi)_{\Omega_\mathrm{f}}| + |(\mathbf{u}_{h,\mathrm{m}}\cdot\nabla\theta,\psi)_{\Omega_\mathrm{m}}| + \frac{1}{2}|(\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}},\theta\psi)_{\Omega_\mathrm{f}}| \\ &\leq \|\mathbf{u}_{h,\mathrm{f}}\|_{0,4,\Omega_\mathrm{f}}\,|\theta|_{1,\Omega_\mathrm{f}}\,\|\psi\|_{0,4,\Omega_\mathrm{f}} + \|\mathbf{u}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_\mathrm{m}}\,\|\theta\|_{1,4,\Omega_\mathrm{m}}\,\|\psi\|_{0,4,\Omega_\mathrm{m}} \\ &\quad + \frac{1}{2}\|\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}\|_{0,\Omega_\mathrm{f}}\|\theta\|_{0,4,\Omega_\mathrm{f}}\|\psi\|_{0,4,\Omega_\mathrm{f}}. \end{aligned}$$

Then, applying the Sobolev inequality (1.7) to $\|\mathbf{u}_{h,\mathrm{f}}\|_{0,4,\Omega_\mathrm{f}}$, $\|\theta\|_{0,4,\Omega_\mathrm{f}}$, $\|\psi\|_{0,4,\Omega_\mathrm{f}}$ and $\|\psi\|_{0,4,\Omega_\mathrm{m}}$, we easily deduce (3.104). $\square$

The following preliminary estimate is an intermediate step to obtain the desired convergence result.

Lemma 3.4.2. *Let us assume that the hypotheses of Theorem 3.2.11 hold and let* $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1) \in \mathbf{H}\times\mathrm{Q}\times\mathrm{H}^1(\Omega)$ *be a solution to* (3.2), *with* $\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$ *and* $\theta_0 \in \mathrm{H}_0^1(\Omega)$. *In addition, we assume that the hypotheses of Theorem 3.3.6 and let* $(\mathbf{u}_h,(p_h,\lambda_h),\theta_h) = (\mathbf{u}_h,(p_h,\lambda_h),\theta_{h,0}+\theta_{h,1}) \in \mathbf{H}_h\times\mathrm{Q}_h\times\Psi_h$ *be a solution to* (3.66), *with* $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_\mathrm{D}) \in \Psi_h$ *and* $\theta_{h,0}\in\Psi_{h,0}$. *Finally, let* $\gamma_\mathbf{g} = \|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}$ *and assume that* $\theta|_{\Omega_\mathrm{m}} \in \mathrm{W}^{1,4}(\Omega_\mathrm{m})$ *holds. There hold*

$$\alpha_\mathrm{F}\,\|\boldsymbol{\chi}_\mathbf{u}\|_\mathbf{H} \leq \widehat{C}_{O_\mathrm{F}}\,\|\boldsymbol{\chi}_{\mathbf{u}_\mathrm{f}}\|_{1,\Omega_\mathrm{f}}\,\|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} + C_D\,\gamma_\mathbf{g}\,\|\chi_\theta\|_{1,\Omega} + L_1, \tag{3.105}$$

$$\alpha_\mathrm{T}\,\|\chi_\theta\|_{1,\Omega} \leq \widehat{C}_{O_\mathrm{T}^3}\,\|\boldsymbol{\chi}_\mathbf{u}\|_\mathbf{H}\,(\|\theta\|_{1,\Omega_\mathrm{f}} + \|\theta\|_{1,4,\Omega_\mathrm{m}}) + L_2, \tag{3.106}$$

and

$$\begin{aligned} \widehat{\beta}\|(\chi_p,\chi_\lambda)\|_\mathrm{Q} &\leq C_{A_\mathrm{F}}\|\mathbf{e}_\mathbf{u}\|_\mathbf{H} + \widehat{C}_{O_\mathrm{F}}\|\mathbf{e}_{\mathbf{u}_\mathrm{f}}\|_{1,\Omega_\mathrm{f}}\|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \\ &\quad + \widehat{C}_{O_\mathrm{F}}\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}}\|\mathbf{e}_{\mathbf{u}_\mathrm{f}}\|_{1,\Omega_\mathrm{f}} + C_D\gamma_\mathbf{g}\|\mathrm{e}_\theta\|_{1,\Omega} + \|(\varrho_p,\varrho_\lambda)\|_\mathrm{Q}, \end{aligned} \tag{3.107}$$

where L_1 *and* L_2 *are defined later in* (3.109) *and* (3.110) *which depend on the solutions* $\mathbf{u}$, $\mathbf{u}_h$ *and* θ.

Proof. Using the definition of the errors given in (3.99), employing (3.103) and subtracting (3.2) and (3.66), we readily obtain

$$\begin{aligned} A_\mathrm{F}(\mathbf{e}_\mathbf{u},\mathbf{v}) + \big[O_\mathrm{F}^h(\mathbf{u}_\mathrm{f};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) - O_\mathrm{F}^h(\mathbf{u}_{h,\mathrm{f}};\mathbf{u}_{h,\mathrm{f}},\mathbf{v}_\mathrm{f})\big] - D(\mathrm{e}_\theta,\mathbf{v}) + B(\mathbf{v},(\varrho_p,\varrho_\lambda)) &= 0, \\ B(\mathbf{e}_\mathbf{u},(q,\xi)) &= 0, \\ A_\mathrm{T}(\mathrm{e}_\theta,\psi) + \big[O_\mathrm{T}^h(\mathbf{u};\theta,\psi) - O_\mathrm{T}^h(\mathbf{u}_h;\theta_h,\psi)\big] &= 0, \end{aligned} \tag{3.108}$$

for all $\mathbf{v} \in \mathbf{V}_h$, $(q,\xi) \in \mathrm{Q}_h$ and $\psi \in \Psi_{h,0}$. Then, adding and subtracting suitable terms and employing the decompositions (3.100) and (3.101), the first equation in (3.108) can be rewritten as

$$A_{\mathrm{F}}(\boldsymbol{\chi}_{\mathbf{u}},\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{u}_{h,\mathrm{f}};\boldsymbol{\chi}_{\mathbf{u}_\mathrm{f}},\mathbf{v}_\mathrm{f}) = -O_{\mathrm{F}}^h(\boldsymbol{\chi}_{\mathbf{u}_\mathrm{f}};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) + D(\chi_\theta,\mathbf{v}) - A_{\mathrm{F}}(\boldsymbol{\varrho}_{\mathbf{u}},\mathbf{v})$$
$$-O_{\mathrm{F}}^h(\mathbf{u}_{h,\mathrm{f}};\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{f}},\mathbf{v}_\mathrm{f}) - O_{\mathrm{F}}^h(\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{f}};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) + D(\varrho_\theta,\mathbf{v}) - B(\mathbf{v},(\varrho_p,\varrho_\lambda)),$$

for all $\mathbf{v} \in \mathbf{V}_h$. In particular, for $\mathbf{v} = \boldsymbol{\chi}_{\mathbf{u}}$, employing the ellipticity of the bilinear form $A_{\mathrm{F}}(\cdot,\cdot) + O_{\mathrm{F}}^h(\mathbf{u}_{h,\mathrm{f}};\cdot,\cdot)$, and the continuity of A_{F}, O_{F}^h, and D (cf. (3.4), (3.67) and (3.6), respectively), we obtain (3.105) where

$$L_1 := C_{A_\mathrm{F}}\,\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathbf{H}} + \widehat{C}_{O_\mathrm{F}}\,(\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} + \|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}})\,\|\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{f}}\|_{1,\Omega_\mathrm{f}}$$
$$+ C_D\,\gamma_{\mathbf{g}}\,\|\varrho_\theta\|_{1,\Omega} + C_B\|(\varrho_p,\varrho_\lambda)\|_{\mathrm{Q}}. \tag{3.109}$$

On the other hand, from the third equation of (3.108), after a simple computations it can be obtained the identity

$$A_{\mathrm{T}}(\chi_\theta,\psi) + O_{\mathrm{T}}^h(\mathbf{u}_h;\chi_\theta,\psi) = -A_{\mathrm{T}}(\varrho_\theta,\psi) - O_{\mathrm{T}}^h(\boldsymbol{\varrho}_{\mathbf{u}};\theta,\psi)$$
$$- O_{\mathrm{T}}^h(\boldsymbol{\chi}_{\mathbf{u}};\theta,\psi) - O_{\mathrm{T}}^h(\mathbf{u}_h;\varrho_\theta,\psi),$$

for all $\psi \in \Psi_{h,0}$. Then, noticing that $\chi_\theta \in \Psi_{h,0}$, we take $\psi = \chi_\theta$ in the latter identity, employ the ellipticity of the bilinear form $A_{\mathrm{T}}(\cdot,\cdot)+O_{\mathrm{T}}^h(\mathbf{u}_h;\cdot,\cdot)$ (cf. (3.72)), the continuity of A_{T} (cf. (3.5)), and (3.104), to obtain (3.106) with

$$L_2 := C_{A_\mathrm{T}}\,\|\varrho_\theta\|_{1,\Omega} + \widehat{C}_{O_\mathrm{T}^3}\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathbf{H}}\,(\|\theta\|_{1,\Omega_\mathrm{f}} + \|\theta\|_{1,4,\Omega_\mathrm{m}})$$
$$+\widehat{C}_{O_\mathrm{T}^3}\|\mathbf{u}_h\|_{\mathbf{H}}\,(\|\varrho_\theta\|_{1,\Omega_\mathrm{f}} + \|\varrho_\theta\|_{1,4,\Omega_\mathrm{m}})\,. \tag{3.110}$$

Now, to estimate χ_p and χ_λ we observe that from the discrete inf-sup condition (3.73), there holds

$$\begin{aligned}
\widehat{\beta}\,\|(\chi_p,\chi_\lambda)\|_{\mathrm{Q}} &\le \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(\chi_p,\chi_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} \\
&\le \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(\mathrm{e}_p,\mathrm{e}_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} + \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},-(\varrho_p,\varrho_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} \\
&\le \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(\mathrm{e}_p,\mathrm{e}_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} + \|(\varrho_p,\varrho_\lambda)\|_{\mathrm{Q}}.
\end{aligned} \tag{3.111}$$

In turn, from the first equations of (3.108), adding and subtracting suitable terms, we obtain

$$\begin{aligned}
B(\mathbf{v},(\mathrm{e}_p,\mathrm{e}_\lambda)) &= -[\,A_{\mathrm{F}}(\mathbf{e}_{\mathbf{u}},\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{u}_\mathrm{f};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) - O_{\mathrm{F}}^h(\mathbf{u}_{h,\mathrm{f}};\mathbf{u}_{h,\mathrm{f}},\mathbf{v}_\mathrm{f}) - D(\mathrm{e}_\theta,\mathbf{v})] \\
&= -[\,A_{\mathrm{F}}(\mathbf{e}_{\mathbf{u}},\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{e}_{\mathbf{u}_\mathrm{f}};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) + O_{\mathrm{F}}^h(\mathbf{u}_{h,\mathrm{f}};\mathbf{e}_{\mathbf{u}_\mathrm{f}},\mathbf{v}_\mathrm{f}) - D(\mathrm{e}_\theta,\mathbf{v})],
\end{aligned}$$

for all $\mathbf{v} \in \mathbf{H}_h$. Then, utilizing the last identity, from (3.111) and the continuity of the forms involved, we obtain

$$\widehat{\beta}\,\|(\chi_p,\chi_\lambda)\|_{\mathrm{Q}} \,\leq\, C_{A_{\mathrm{F}}}\|\mathbf{e_u}\|_{\mathbf{H}} \,+\, \widehat{C}_{O_{\mathrm{F}}}\|\mathbf{e}_{\mathbf{u}_{\mathrm{f}}}\|_{1,\Omega_{\mathrm{f}}}\|\mathbf{u}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \,+\, \widehat{C}_{O_{\mathrm{F}}}\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}\|\mathbf{e}_{\mathbf{u}_{\mathrm{f}}}\|_{1,\Omega_{\mathrm{f}}} \\ + C_D\gamma_{\mathbf{g}}\|\mathrm{e}_\theta\|_{1,\Omega} \,+\, \|(\varrho_p,\varrho_\lambda)\|_{\mathrm{Q}}.$$

□

Now we are in position of establishing the Cea's estimate for the Galerkin scheme. This is established next.

Theorem 3.4.3. *Let us assume that the hypotheses of Lemma 3.4.2 hold and let* $(\mathbf{u},(p,\lambda),\theta) = ((\mathbf{u}_{\mathrm{f}},\mathbf{u}_{\mathrm{m}}),(p,\lambda),\theta_0+\theta_1) \in \mathbf{H}\times\mathrm{Q}\times\mathrm{H}^1(\Omega)$ *be the unique solution of* (3.2) *and* $(\mathbf{u}_h,(p_h,\lambda_h),\theta_h) = ((\mathbf{u}_{h,\mathrm{f}},\mathbf{u}_{h,\mathrm{m}}),(p_h,\lambda_h),\theta_{h,0}+\theta_{h,1}) \in \mathbf{H}_h\times\mathrm{Q}_h\times\Psi_h$ *be a solution to* (3.66). *Finally, let* $\gamma_{\mathbf{g}} = \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$ *and assume further that*

$$C_1\gamma_{\mathbf{g}}\delta^{-3/4}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \,+\, C_2\gamma_{\mathbf{g}}\|\theta\|_{1,4,\Omega_{\mathrm{m}}} \leq \frac{1}{2}, \tag{3.112}$$

with $C_1\,,C_2>0$, *independent of* h, δ, $\gamma_{\mathbf{g}}$ *and* θ_{D}. *There exists* $C>0$, *independent of the aforementioned datum, such that*

$$\begin{aligned}\|\mathbf{e_u}\|_{\mathbf{H}} + \|(\mathrm{e}_p,\mathrm{e}_\lambda)\|_{\mathrm{Q}} + \|\mathrm{e}_\theta\|_{1,\Omega} \leq C\Bigg\{ &\inf_{\mathbf{v}_{h,\mathrm{f}}\in\mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})} \|\mathbf{u}_{\mathrm{f}}-\mathbf{v}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \\ + \inf_{\mathbf{v}_{h,\mathrm{m}}\in\mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})} \|\mathbf{u}_{\mathrm{m}}-\mathbf{v}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} &+ \inf_{q_h\in\mathrm{L}_{h,0}(\Omega)} \|p-q_h\|_{0,\Omega} \\ + \inf_{\xi_h\in\Lambda_h(\Sigma)} \|\lambda-\xi_h\|_{1/2,\Sigma} + \inf_{\psi_h\in\Psi_h^{\Gamma}} &\big(\|\theta-\psi_h\|_{1,\Omega_{\mathrm{f}}} + \|\theta-\psi_h\|_{1,4,\Omega_{\mathrm{m}}}\big)\Bigg\}.\end{aligned} \tag{3.113}$$

Proof. Let us first recall that from (3.36), (3.40) and (3.23b) with $\zeta = \theta_{\mathrm{D}}$, the following estimates hold

$$\begin{aligned}\|\mathbf{u}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} &\leq C_{\mathbf{u}}\gamma_{\mathbf{g}}C_{\mathrm{lift},2}\,\delta^{-3/4}(1+\delta^2)^{1/2}\,\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}, \\ \|\theta\|_{1,\Omega_{\mathrm{f}}} &\leq (C_\theta+1)C_{\mathrm{lift},2}\,\delta^{-3/4}(1+\delta^2)^{1/2}\,\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}.\end{aligned} \tag{3.114}$$

Then, combining (3.105), (3.106), and (3.114), we obtain

$$\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{H}} \leq \big(C_1\gamma_{\mathbf{g}}\delta^{-3/4}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \,+\, C_2\gamma_{\mathbf{g}}\|\theta\|_{1,4,\Omega_{\mathrm{m}}}\big)\,\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{H}} \\ + \alpha_{\mathrm{F}}^{-1}\alpha_{\mathrm{T}}^{-1}C_D\gamma_{\mathbf{g}}L_2 \,+\, \alpha_{\mathrm{F}}^{-1}L_1,$$

with

$$C_1 := \alpha_{\mathrm{F}}^{-1} C_{\mathrm{lift},2} \left(\widehat{C}_{O_{\mathrm{F}}} C_{\mathbf{u}} + \alpha_{\mathrm{T}}^{-1} C_D \widehat{C}_{O_{\mathrm{T}}^3} (C_\theta + 1) \right) \quad \text{and} \quad C_2 := \alpha_{\mathrm{F}}^{-1} \alpha_{\mathrm{T}}^{-1} C_D \widehat{C}_{O_{\mathrm{T}}^3},$$

which together with (3.112), implies that

$$\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{H}} \leq 2\alpha_{\mathrm{F}}^{-1} \alpha_{\mathrm{T}}^{-1} C_D \gamma_{\mathbf{g}} L_2 + 2\alpha_{\mathrm{F}}^{-1} L_1. \tag{3.115}$$

Now, to estimate L_1 and L_2 we recall that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$ (cf. (3.94)) which yields $\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \leq \|\mathbf{u}_h\|_{\mathbf{H}} \leq C_{\mathbf{u}} \gamma_{\mathbf{g}} \|\theta_{h,1}\|_{1,\Omega}$, which combined with (3.89b), implies

$$\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \leq \|\mathbf{u}_h\|_{\mathbf{H}} \leq C_{\mathbf{u}} \gamma_{\mathbf{g}} \widehat{C}_{\mathrm{lift},2}\, \delta^{1/4} (2 + \delta^{-1})\, \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}. \tag{3.116}$$

Then, from the latter, the definition of L_1 and L_2, we get

$$L_1 \leq c_1 \left(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathbf{H}} + \|\varrho_\theta\|_{1,\Omega} + \|(\varrho_p, \varrho_\lambda)\|_{\mathrm{Q}} \right), \tag{3.117}$$

$$L_2 \leq c_2 \left(\|\varrho_\theta\|_{1,\Omega} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}} + \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathbf{H}} \right), \tag{3.118}$$

with c_1, c_2 being positive constants independent of h. In this way, using (3.115), (3.117), (3.118) and the fact that $\mathrm{W}^{1,4}(\Omega_{\mathrm{m}})$ is continuously embedded in $\mathrm{H}^1(\Omega_{\mathrm{m}})$, which implies $\|\varrho_\theta\|_{1,\Omega} \leq c(\|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}})$, we readily obtain

$$\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{H}} \leq c_3 \left(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathbf{H}} + \|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}} \right) + c_4 \|(\varrho_p, \varrho_\lambda)\|_{\mathrm{Q}}, \tag{3.119}$$

with c_3, $c_4 > 0$, independent of h.

Now, to estimate $\|\chi_\theta\|_{1,\Omega}$ we simply substitute (3.119) in (3.106) and employ (3.112), the second estimate in (3.114) and (3.118), to obtain

$$\|\chi_\theta\|_{1,\Omega} \leq c_5 \left(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathbf{H}} + \|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}} + \|(\varrho_p, \varrho_\lambda)\|_{\mathrm{Q}} \right), \tag{3.120}$$

with $c_5 > 0$, independent of h.

According to the above, from (3.100), (3.119), (3.120), the triangle inequality, the fact that $\widehat{\mathbf{v}}_h = (\widehat{\mathbf{v}}_{h,\mathrm{f}}, \widehat{\mathbf{v}}_{h,\mathrm{m}}) \in \mathbf{V}_h$, $(\widehat{q}_h, \widehat{\xi}_h) \in \mathrm{Q}_h$, and $\widehat{\psi}_h \in \Psi_h^\Gamma$ are arbitrary and (3.102), we obtain

$$\begin{aligned} \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{H}} + \|\mathrm{e}_\theta\|_{1,\Omega} \leq c_6 \Bigg\{ & \inf_{\mathbf{v}_{h,\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})} \|\mathbf{u}_{\mathrm{f}} - \mathbf{v}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \\ & + \inf_{\mathbf{v}_{h,\mathrm{m}} \in \mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})} \|\mathbf{u}_{\mathrm{m}} - \mathbf{v}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} + \inf_{q_h \in \mathrm{L}_{h,0}(\Omega)} \|p - q_h\|_{0,\Omega} \\ & + \inf_{\xi_h \in \Lambda_h(\Sigma)} \|\lambda - \xi_h\|_{1/2,\Sigma} + \inf_{\psi_h \in \Psi_h^\Gamma} \left(\|\theta - \psi_h\|_{1,\Omega_{\mathrm{f}}} + \|\theta - \psi_h\|_{1,4,\Omega_{\mathrm{m}}} \right) \Bigg\}, \end{aligned} \tag{3.121}$$

with $c_6 > 0$, independent of h.

Finally, and similarly to the above, from (3.107), the first inequality in (3.114), (3.116), (3.121) and the triangle inequality, we deduce that there exists $c_7 > 0$, independent of h, such that

$$\begin{aligned}\|(\mathrm{e}_p, \mathrm{e}_\lambda)\|_{\mathrm{Q}} \;&\leq\; c_7 \Bigg\{ \inf_{\mathbf{v}_{h,\mathrm{f}} \in \mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f})} \|\mathbf{u}_\mathrm{f} - \mathbf{v}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} \\ &+ \inf_{\mathbf{v}_{h,\mathrm{m}} \in \mathbf{H}_{h,\Gamma_\mathrm{m}}(\Omega_\mathrm{m})} \|\mathbf{u}_\mathrm{m} - \mathbf{v}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_\mathrm{m}} + \inf_{q_h \in \mathrm{L}_{h,0}(\Omega)} \|p - q_h\|_{0,\Omega} \\ &+ \inf_{\xi_h \in \Lambda_h(\Sigma)} \|\lambda - \xi_h\|_{1/2,\Sigma} + \inf_{\psi_h \in \Psi_h^\Gamma} \Big(\|\theta - \psi_h\|_{1,\Omega_\mathrm{f}} + \|\theta - \psi_h\|_{1,4,\Omega_\mathrm{m}} \Big) \Bigg\},\end{aligned}$$

which together with (3.121) implies (3.113) and concludes the proof. □

We conclude this section by deriving the theoretical rate of convergence for the Galerkin scheme (3.66). To that end we recall that the discrete spaces satisfy the following approximation properties (see [11, 18, 47, 52]):

($\mathbf{AP}_h^{\mathbf{u}_\mathrm{f}}$) For each $\mathbf{v}_\mathrm{f} \in \mathbf{H}^2(\Omega_\mathrm{f}) \cap \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, there exists $\mathbf{v}_{h,\mathrm{f}} \in \mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, such that

$$\|\mathbf{v}_\mathrm{f} - \mathbf{v}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} \;\leq\; Ch\, \|\mathbf{v}_\mathrm{f}\|_{2,\Omega_\mathrm{f}}.$$

($\mathbf{AP}_h^{\mathbf{u}_\mathrm{m}}$) For each $\mathbf{v}_\mathrm{m} \in \mathbf{H}^1(\Omega_\mathrm{m}) \cap \mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m})$, with $\mathrm{div}\,\mathbf{v}_\mathrm{m} \in \mathrm{H}^1(\Omega_\mathrm{m})$, there exists $\mathbf{v}_{h,\mathrm{m}} \in \mathbf{H}_{h,\Gamma_\mathrm{m}}(\Omega_\mathrm{m})$, such that

$$\|\mathbf{v}_\mathrm{m} - \mathbf{v}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_\mathrm{m}} \;\leq\; Ch\, \left\{\|\mathbf{v}_\mathrm{m}\|_{1,\Omega_\mathrm{m}} \,+\, \|\mathrm{div}\,\mathbf{v}_\mathrm{m}\|_{1,\Omega_\mathrm{m}}\right\}.$$

($\mathbf{AP}_h^{p}$) For each $q \in \mathrm{H}^1(\Omega) \cap \mathrm{L}^2_0(\Omega)$, there exists $q_h \in \mathrm{L}_{h,0}(\Omega)$ such that

$$\|q - q_h\|_{0,\Omega} \;\leq\; Ch\, \|q\|_{1,\Omega}.$$

($\mathbf{AP}_h^{\lambda}$) For each $\xi \in \mathrm{W}^{3/2,2}(\Sigma)$, there exists $\xi_h \in \Lambda_h(\Sigma)$ such that

$$\|\xi - \xi_h\|_{1/2,\Sigma} \;\leq\; Ch\, \|\xi\|_{3/2,\Sigma}.$$

($\mathbf{AP}_h^{\theta}$) For each $\psi \in \mathrm{H}^2(\Omega)$, with $\psi|_\Gamma = \theta_\mathrm{D}$ and $\psi|_{\Omega_\mathrm{m}} \in \mathrm{W}^{2,4}(\Omega_\mathrm{m})$, there exists $\psi_h \in \Psi_h^\Gamma$, such that

$$\|\psi - \psi_h\|_{1,\Omega_\mathrm{f}} + \|\psi - \psi_h\|_{1,4,\Omega_\mathrm{m}} \;\leq\; Ch\{\|\psi\|_{2,\Omega_\mathrm{f}} \,+\, \|\psi\|_{2,4,\Omega_\mathrm{m}}\}.$$

Using this approximation properties we obtain the following result.

Theorem 3.4.4. *Assume the same hypotheses in Theorem 3.4.3 hold, let* $(\mathbf{u},(p,\lambda),\theta) = ((\mathbf{u}_{\mathrm{f}},\mathbf{u}_{\mathrm{m}}),(p,\lambda),\theta_0+\theta_1) \in \mathbf{H}\times \mathrm{Q}\times \mathrm{H}^1(\Omega)$ *be the unique solution of* (3.2) *and* $(\mathbf{u}_h,(p_h,\lambda_h),\theta_h) = ((\mathbf{u}_{h,\mathrm{f}},\mathbf{u}_{h,\mathrm{m}}),(p_h,\lambda_h),\theta_{h,0}+\theta_{h,1}) \in \mathbf{H}_h\times \mathrm{Q}_h\times \Psi_h$ *be a solution to* (3.66). *Assume further that* $\mathbf{u}_{\mathrm{f}}\in \mathbf{H}^2(\Omega_{\mathrm{f}})$, $\mathbf{u}_{\mathrm{m}}\in \mathbf{H}^1(\Omega_{\mathrm{m}})$, $\operatorname{div}\mathbf{u}_{\mathrm{m}}\in \mathrm{H}^1(\Omega_{\mathrm{m}})$, $p\in \mathrm{H}^1(\Omega)$, $\lambda\in \mathrm{H}^{3/2}(\Sigma)$, $\theta|_{\Omega_{\mathrm{f}}}\in \mathrm{H}^2(\Omega_{\mathrm{f}})$, *and* $\theta|_{\Omega_{\mathrm{m}}}\in \mathrm{W}^{2,4}(\Omega_{\mathrm{m}})$. *Then, there exists* $C_{rate}>0$, *independent of* h *and the continuous and discrete solutions, such that*

$$\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{H}} + \|(\mathrm{e}_p,\mathrm{e}_\lambda)\|_{\mathrm{Q}} + \|\mathrm{e}_\theta\|_{1,\Omega} \le C_{rate} h\left\{\|\mathbf{u}_{\mathrm{f}}\|_{2,\Omega_{\mathrm{f}}} + \|\mathbf{u}_{\mathrm{m}}\|_{1,\Omega_{\mathrm{m}}} + \|\operatorname{div}\mathbf{u}_{\mathrm{m}}\|_{1,\Omega_{\mathrm{m}}} + \|p\|_{1,\Omega} + \|\lambda\|_{3/2,\Sigma} + \|\theta\|_{2,\Omega_{\mathrm{f}}} + \|\theta\|_{2,4,\Omega_{\mathrm{m}}}\right\}.$$

Proof. The result is a direct consequence of Theorem 3.4.3 and the approximation properties $(\mathbf{AP}_h^{\mathbf{u}_{\mathrm{f}}})$, $(\mathbf{AP}_h^{\mathbf{u}_{\mathrm{m}}})$, $(\mathbf{AP}_h^{p})$, $(\mathbf{AP}_h^{\lambda})$ and $(\mathbf{AP}_h^{\theta})$. □

3.5 Numerical results

In this section we restrict ourselves to the 2D case and present some numerical results illustrating the performance of our finite element scheme (3.66) on a set of quasi-uniform triangulations of the corresponding domains and considering the finite element spaces introduced in Section 3.3. Our implementation is based on a *FreeFem++* code [68], in conjunction with the direct linear solver *UMFPACK* [41]. In order to solve the nonlinear problem, we propose the Newton-type strategy: Starting with the initial guess $\mathbf{u}^0 = (\mathbf{u}_{\mathrm{f}}^0,\mathbf{u}_{\mathrm{m}}^0)\in \mathbf{H}_h$ and $\theta^0\in\Psi_h$, for $n\ge 1$, find $\mathbf{u}^n\in\mathbf{H}_h$, $(p^n,\lambda^n)\in \mathrm{Q}_h$, and $\theta^n\in\Psi_h$, such that $\theta_h|_\Gamma = \theta^\delta_{\mathrm{D},h}$ and

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u}^n,\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{u}_{\mathrm{f}}^{n-1};\mathbf{u}_{\mathrm{f}}^n,\mathbf{v}_{\mathrm{f}}) + O_{\mathrm{F}}^h(\mathbf{u}_{\mathrm{f}}^n;\mathbf{u}_{\mathrm{f}}^{n-1},\mathbf{v}_{\mathrm{f}}) & \\ +B(\mathbf{v},(p^n,\lambda^n)) - D(\theta^n,\mathbf{v}) &= O_{\mathrm{F}}^h(\mathbf{u}_{\mathrm{f}}^{n-1};\mathbf{u}_{\mathrm{f}}^{n-1},\mathbf{v}_{\mathrm{f}}),\\ B(\mathbf{u}^n,(q,\xi)) &= 0,\\ A_{\mathrm{T}}(\theta^n,\psi) + O_{\mathrm{T}}^h(\mathbf{u}^{n-1};\theta^n,\psi) + O_{\mathrm{T}}^h(\mathbf{u}^n;\theta^{n-1},\psi) &= O_{\mathrm{T}}^h(\mathbf{u}^{n-1};\theta^{n-1},\psi), \end{aligned}$$

for all $\mathbf{v} = (\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{m}})\in\mathbf{H}_h$, $(q,\xi)\in \mathrm{Q}_h$, and $\psi\in\Psi_h$.

We note here that $\theta^\delta_{\mathrm{D},h}$ coincides with the nodal interpolation of θ_{D} on Γ. Therefore, in practice, there is no need to construct the operator $\mathrm{E}_{\delta,h}$ for the computational implementation of the method.

The iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, that is

$$\frac{\|\mathbf{coeff}^{n+1} - \mathbf{coeff}^{n}\|_{l^2}}{\|\mathbf{coeff}^{n+1}\|_{l^2}} \le tol,$$

where $\|\cdot\|_{l^2}$ stands for the usual euclidean norm in $\mathbb{R}^{\text{dof}}$, with dof denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h(\Omega_{\rm f})$, $\mathbf{H}_h(\Omega_{\rm m})$, $\mathrm{L}_h(\Omega)$, $\Lambda_h(\Sigma)$, and Ψ_h, and tol is a fixed tolerance. For each example shown below we simply take $\mathbf{u}^0 = \mathbf{0}$ and $\theta^0 = 0$ as initial guess and $tol = 1e-6$.

Now, we introduce some additional notations. We denote by $h_\Sigma := \max\{h_F : F \in \Sigma_{2h}\}$. As in Section 3.4, the individual errors for each variable are denoted by $\mathbf{e}_{\mathbf{u}_{\rm f}}$, $\mathbf{e}_{\mathbf{u}_{\rm m}}$, e_λ, e_p and e_θ and let $\mathrm{e}_{p_\star} = \mathrm{e}_p|_{\Omega_\star}$, $\mathrm{e}_{\theta_\star} = \mathrm{e}_\theta|_{\Omega_\star}$, for $\star \in \{\mathrm{f}, \mathrm{m}\}$. In addition, we define the experimental rates of convergence $\mathbf{r}_{\mathbf{u}_{\rm f}}$, $\mathbf{r}_{\mathbf{u}_{\rm m}}$, $\mathrm{r}_{p_{\rm f}}$, $\mathrm{r}_{p_{\rm m}}$, r_λ, $\mathrm{r}_{\theta_{\rm f}}$ and $\mathrm{r}_{\theta_{\rm m}}$, as

$$\mathbf{r}_{\mathbf{u}_{\rm f}} := \frac{\log(\mathbf{e}_{\mathbf{u}_{\rm f}}/\mathbf{e}'_{\mathbf{u}_{\rm f}})}{\log(h_{\rm f}/h'_{\rm f})}, \qquad \mathbf{r}_{\mathbf{u}_{\rm m}} := \frac{\log(\mathbf{e}_{\mathbf{u}_{\rm m}}/\mathbf{e}'_{\mathbf{u}_{\rm m}})}{\log(h_{\rm m}/h'_{\rm m})},$$

$$\mathrm{r}_{p_{\rm f}} := \frac{\log(\mathrm{e}_{p_{\rm f}}/\mathrm{e}'_{p_{\rm f}})}{\log(h_{\rm f}/h'_{\rm f})}, \qquad \mathrm{r}_{p_{\rm m}} := \frac{\log(\mathrm{e}_{p_{\rm m}}/\mathrm{e}'_{p_{\rm m}})}{\log(h_{\rm m}/h'_{\rm m})},$$

$$\mathrm{r}_\lambda := \frac{\log(\mathrm{e}_\lambda/\mathrm{e}'_\lambda)}{\log(h_\Sigma/h'_\Sigma)}, \quad \mathrm{r}_{\theta_{\rm f}} := \frac{\log(\mathrm{e}_{\theta_{\rm f}}/\mathrm{e}'_{\theta_{\rm f}})}{\log(h_{\rm f}/h'_{\rm f})}, \quad \mathrm{r}_{\theta_{\rm m}} := \frac{\log(\mathrm{e}_{\theta_{\rm m}}/\mathrm{e}'_{\theta_{\rm m}})}{\log(h_{\rm m}/h'_{\rm m})},$$

where $h_\star$ and $h'_\star$ ($\star \in \{\mathrm{f}, \mathrm{m}, \Sigma\}$) denote two consecutive mesh sizes with their respective errors $\mathbf{e}$, $\mathbf{e}'$ (or e, e').

Example 1: Manufactured Exact Solution

In our first example we illustrate the accuracy of our method considering a manufactured exact solution defined on $\Omega = \Omega_{\rm f} \cup \Sigma \cup \Omega_{\rm m}$, with $\Omega_{\rm f} := (-1/2, 1/2) \times (0, 1/2)$ and $\Omega_{\rm m} := (-1/2, 1/2) \times (-1/2, 0)$. We consider the following parameters $\mu = 1$, $\mathbf{g}_{\rm f} = (0, -1)^t$, $\mathbf{g}_{\rm m} = (0, -1)^t$, $\omega_1 = 1$, $\kappa_{\rm f} = 1$, $\kappa_{\rm m} = 1$ and $\mathbf{K} = \mathbf{I}$, and the terms on the right-hand side are adjusted so that the exact solution is given by the functions

$$\mathbf{u}_{\rm f}(x, y) := \begin{pmatrix} 16y\cos(\pi x)^2(y^2 - 1/4) \\ 8\pi\cos(\pi x)\sin(\pi x)(y^2 - 1/4)^2 \end{pmatrix} \quad \text{in} \quad \Omega_{\rm f},$$

$$\mathbf{u}_{\rm m}(x, y) := \begin{pmatrix} -2y\cos(\pi x)^2 \\ -2\pi\cos(\pi x)\sin(\pi x)(y^2 - 1/4) \end{pmatrix} \quad \text{in} \quad \Omega_{\rm m},$$

$$p_\star(x, y) := \exp(y)\sin(x) \quad \text{in} \quad \Omega_\star,$$

$$\theta_\star(x, y) := \exp(-xy) \quad \text{in} \quad \Omega_\star,$$

with $\star \in \{\mathrm{f}, \mathrm{m}\}$. We notice that $\mathbf{u}_{\rm f}|_\Sigma = \mathbf{u}_{\rm m}|_\Sigma$, $\theta_{\rm f}|_\Sigma = \theta_{\rm m}|_\Sigma$, and $\kappa_{\rm f}\nabla\theta_{\rm f}|_\Sigma = \kappa_{\rm m}\nabla\theta_{\rm m}|_\Sigma$. We notice also that these functions do not satisfy the interface conditions (2.5), thus the difference must be incorporated as a functional at the right-hand side of the resulting system.

In Table 3.1 we summarize the history of convergence for a sequence of quasi-uniform triangulations. We observe there that the rate of convergence $O(h)$ predicted by Theorem 3.4.4 is attained in all the cases.

dof	$h_{\rm f}$	$\mathbf{e}_{\mathbf{u}_{\rm f}}$	$\mathbf{r}_{\mathbf{u}_{\rm f}}$	$\mathrm{e}_{p_{\rm f}}$	$\mathrm{r}_{p_{\rm f}}$	$\mathrm{e}_{\theta_{\rm f}}$	$\mathrm{r}_{\theta_{\rm f}}$
216	0.3207	0.5592	–	0.2104	–	0.0813	–
834	0.1804	0.3492	0.8189	0.1133	1.0767	0.0390	1.2762
3026	0.1013	0.1844	1.1057	0.0565	1.2039	0.0199	1.1657
11738	0.0503	0.0855	1.0989	0.0292	0.9398	0.0099	0.9945
45622	0.0247	0.0424	0.9875	0.0145	0.9926	0.0050	0.9674
180930	0.0123	0.0208	1.0226	0.0070	1.0338	0.0025	0.9818
725890	0.0065	0.0103	1.1014	0.0035	1.1007	0.0012	1.1210

dof	$h_{\rm m}$	$\mathbf{e}_{\mathbf{u}_{\rm m}}$	$\mathbf{r}_{\mathbf{u}_{\rm m}}$	$\mathrm{e}_{p_{\rm m}}$	$\mathrm{r}_{p_{\rm m}}$	$\mathrm{e}_{\theta_{\rm m}}$	$\mathrm{r}_{\theta_{\rm m}}$
216	0.3663	0.1752	–	0.0330	–	0.0953	–
834	0.1804	0.0748	1.2018	0.0137	1.2349	0.0377	1.3073
3026	0.0951	0.0398	0.9856	0.0072	1.0013	0.0200	0.9917
11738	0.0488	0.0198	1.0416	0.0035	1.0802	0.0098	1.0692
45622	0.0247	0.0099	1.0091	0.0017	0.9984	0.0050	0.9764
180930	0.0143	0.0050	1.2604	0.0008	1.2650	0.0025	1.2636
725890	0.0064	0.0024	0.8754	0.0004	0.8686	0.0012	0.8831

dof	h_{Σ}	e_{λ}	r_{λ}	iteration
216	0.2500	0.0470	–	5
834	0.1250	0.0308	0.6107	5
3026	0.0625	0.0114	1.4330	5
11738	0.0312	0.0051	1.1391	5
45622	0.0156	0.0026	0.9634	5
180930	0.0078	0.0013	1.0307	5
725890	0.0039	0.0006	1.0442	5

Table 3.1: EXAMPLE 1: Degree of Freedom, mesh sizes, errors, rates of convergence and number of iterations for the coupled problem.

Example 2: Kovasznay's analytical solution

In our second example, we focus on the performance of the iterative method with respect to the viscosity μ. To this end, we consider the domain $\Omega = \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$, with $\Omega_{\mathrm{f}} := (-1/2, 3/2) \times (0, 1/2)$ and $\Omega_{\mathrm{m}} := (-1/2, 3/2) \times (-1/2, 0)$. In turn, the terms on the right-hand side of (3.2) are adjusted so that the exact solution are given by the functions

$$\begin{aligned} \mathbf{u}_{\mathrm{f}}(x,y) &:= \begin{pmatrix} 1 - e^{\zeta x}\cos(2\pi y) \\ \dfrac{\zeta}{2\pi} e^{\zeta x}\sin(2\pi y) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{f}}, \\ p_{\mathrm{f}}(x,y) &:= -0.5 e^{2\zeta x} + c_0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \\ \mathbf{u}_{\mathrm{m}}(x,y) &:= \begin{pmatrix} (x+0.5)(x-1.5) \\ -(2x-1)(y+2) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{m}}, \\ p_{\mathrm{m}}(x,y) &:= (x-0.5)^3(y-1) \quad \text{in} \quad \Omega_{\mathrm{m}}, \\ \theta(x,y) &:= x^2 + y^2 \quad \text{in} \quad \Omega, \end{aligned}$$

with ζ given by

$$\zeta := \frac{-8\pi^2}{\mu^{-1} + \sqrt{\mu^{-2} + 16\pi^2}},$$

where $\mu > 0$ is the viscosity of the fluid and c_0 is a constant chosen in such a way $(p_{\mathrm{f}}, 1)_{\Omega_{\mathrm{f}}} = 0$. We note that $(\mathbf{u}_{\mathrm{f}}, p_{\mathrm{f}})$ is the well known analytical solution for the Navier–Stokes problem obtained by Kovasznay in [73], which presents a boundary layer at $\{-1/2\} \times (0, 2)$.

In Table 3.2 we display the behavior of the iterative method as a function of the viscosity number, considering different meshsizes and a tolerance $tol = 1e{-6}$. Here we observe that the smaller the parameter μ the higher the number of iterations. In addition, in Table 3.3 we consider $\mu = 0.01$ and observe that, the rates of convergence predicted by the theory, are attained for all the unknowns.

μ	$h = 0.1972$	$h = 0.0993$	$h = 0.0498$	$h = 0.0254$	$h = 0.0130$	$h = 0.0068$
1	5	5	5	5	5	5
0.1	7	7	7	7	7	7
0.01	8	8	8	8	8	8

Table 3.2: Example 2: Number of iterations of the iterative method with respect to μ.

Example 3: Nondimensional problem

In our last example, we are interested in studying the phenomenon on a square cavity with differentially heated walls. To that end, we let $\Omega = \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$, with $\Omega_{\mathrm{f}} := (0,1) \times (0,3/4)$ and $\Omega_{\mathrm{m}} := (0,1) \times (3/4,1)$, and similarly to [79, Section 2.4] we consider the problem with dimensionless numbers

$$
\begin{aligned}
\boldsymbol{\sigma}_{\mathrm{f}} = 2\,\mathbf{e}(\mathbf{u}_{\mathrm{f}}) - p_{\mathrm{f}}\,\mathbf{I} &\quad \text{in} \quad \Omega_{\mathrm{f}},\\
-\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{f}} + (\mathbf{u}_{\mathrm{f}} \cdot \nabla)\,\mathbf{u}_{\mathrm{f}} - \mathrm{Ra}_{\mathrm{f}}\,\mathbf{g}_{\mathrm{f}}\,\theta_{\mathrm{f}} = \mathbf{0} &\quad \text{in} \quad \Omega_{\mathrm{f}},\\
\mathrm{div}\,\mathbf{u}_{\mathrm{f}} = 0 &\quad \text{in} \quad \Omega_{\mathrm{f}},\\
-\Delta\,\theta_{\mathrm{f}} + \mathrm{Pr}_{\mathrm{f}}\,\mathbf{u}_{\mathrm{f}} \cdot \nabla\,\theta_{\mathrm{f}} = 0 &\quad \text{in} \quad \Omega_{\mathrm{f}},\\
\mathbf{u}_{\mathrm{m}} + \mathrm{Da}\,\nabla p_{\mathrm{m}} - \mathrm{Ra}_{\mathrm{m}}\,\mathbf{g}_{\mathrm{m}}\,\theta_{\mathrm{m}} = \mathbf{0} &\quad \text{in} \quad \Omega_{\mathrm{m}},\\
\mathrm{div}\,\mathbf{u}_{\mathrm{m}} = 0 &\quad \text{in} \quad \Omega_{\mathrm{m}},\\
-\Delta\,\theta_{\mathrm{m}} + \mathrm{Pr}_{\mathrm{m}}\,\mathbf{u}_{\mathrm{m}} \cdot \nabla\theta_{\mathrm{m}} = 0 &\quad \text{in} \quad \Omega_{\mathrm{m}},\\
\nabla\,\theta_{\mathrm{f}} \cdot \mathbf{n} = \nabla\,\theta_{\mathrm{m}} \cdot \mathbf{n} &\quad \text{on} \quad \Sigma,\\
\boldsymbol{\sigma}_{\mathrm{f}}\,\mathbf{n} + \frac{\alpha_d}{\sqrt{\mathrm{Da}}}(\mathbf{u}_{\mathrm{f}} \cdot \mathbf{t})\,\mathbf{t} = -p_{\mathrm{m}}\,\mathbf{n} &\quad \text{on} \quad \Sigma,\\
\theta = \theta_{\mathrm{D}} &\quad \text{on} \quad \Gamma,
\end{aligned}
$$

where $\mathrm{Pr}_\star$ and $\mathrm{Ra}_\star$ represent the Prandtl and Rayleigh numbers in the domain $\Omega_\star$ for $\star \in \{\mathrm{f},\mathrm{m}\}$ and Da represents the Darcy number. Here, we fix the Prandtl, Rayleigh and Darcy numbers as $\mathrm{Pr}_{\mathrm{f}} = 0.5$, $\mathrm{Pr}_{\mathrm{m}} = 0.5$, $\mathrm{Ra}_{\mathrm{f}} = 2000$, $\mathrm{Ra}_{\mathrm{m}} = 2000$, and $\mathrm{Da} = 1$, and consider $\alpha_d = 1$. For the boundary condition, we choose $\theta_{\mathrm{D}}(x,y) = 0.5(1-\cos(2\pi x))(1-y)$ on Γ and observe that $\theta_{\mathrm{D}} = 0$ on the left, bottom and right walls whereas on the top wall θ_{D} has a sinusoidal profile with a peak of temperature $\theta_{\mathrm{D}} = 1$ at $x = 0.5$.

In Figure 3.1 we display the approximate solutions obtained with dof = 45726. At the top-left of Figure 3.1 we display the respective mesh with 1378 triangles in Ω_{f} and 584 in Ω_{m}, whereas the velocity vector field, the pressure and temperature are shown at the top-right, bottom-left and bottom-right of Figure 3.1, respectively. There, it is possible to see the expected physical behavior from [40], that is, convection currents form inside the cavity in a symmetric configuration. However, in our case, the interface plays a role in the phenomenon, and as reported in [76], we observe that the velocity field has an expected velocity decrease when it crosses the interface from the free fluid region to the porous medium.

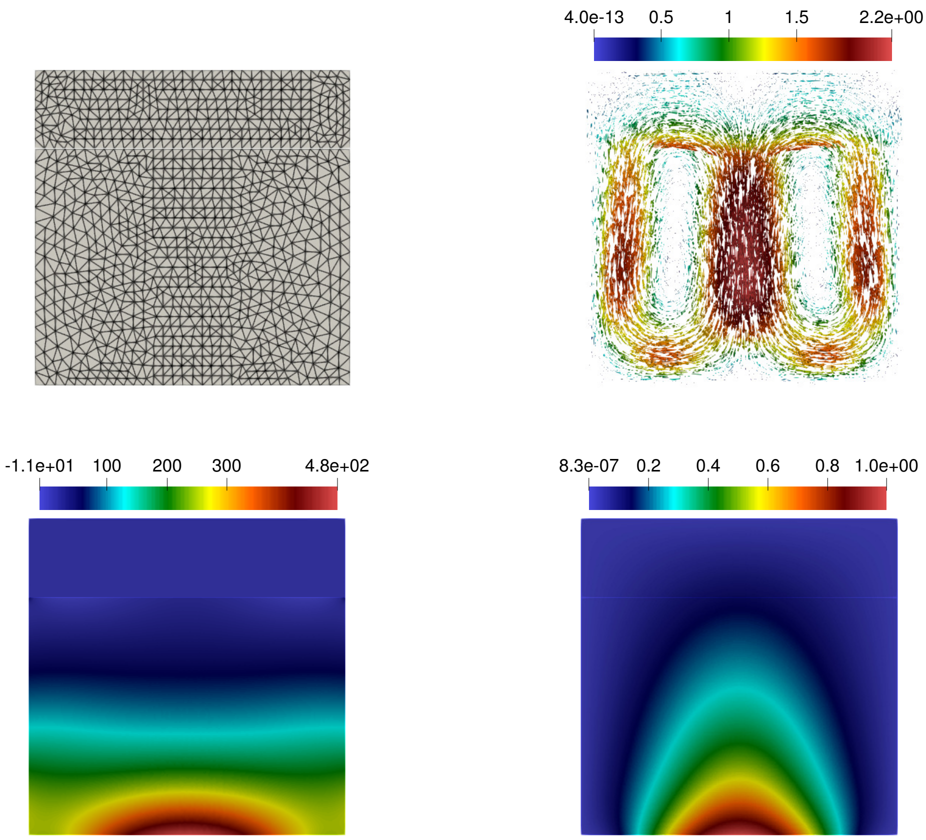

Figure 3.1: EXAMPLE 3: Mesh (top-left), Velocity vector field (top-right), pressure (bottom-left) and temperature (bottom-right).

dof	h_{f}	$\mathbf{e}_{\mathbf{u}_{\mathrm{f}}}$	$\mathbf{r}_{\mathbf{u}_{\mathrm{f}}}$	$\mathrm{e}_{p_{\mathrm{f}}}$	$\mathrm{r}_{p_{\mathrm{f}}}$	$\mathrm{e}_{\theta_{\mathrm{f}}}$	$\mathrm{r}_{\theta_{\mathrm{f}}}$
1762	0.1972	0.5819	–	0.0102	–	0.0828	–
6460	0.0993	0.2774	1.0794	0.0044	1.2025	0.0410	1.0241
25101	0.0498	0.1389	1.0025	0.0021	1.0593	0.0209	0.9744
98357	0.0254	0.0688	1.0475	0.0010	1.0555	0.0105	1.0262
394038	0.0128	0.0342	1.0160	0.0005	1.0189	0.0053	0.9893
1668858	0.0068	0.0160	1.2114	0.0002	1.2329	0.0026	1.1264

dof	h_{m}	$\mathbf{e}_{\mathbf{u}_{\mathrm{m}}}$	$\mathbf{r}_{\mathbf{u}_{\mathrm{m}}}$	$\mathrm{e}_{p_{\mathrm{m}}}$	$\mathrm{r}_{p_{\mathrm{m}}}$	$\mathrm{e}_{\theta_{\mathrm{m}}}$	$\mathrm{r}_{\theta_{\mathrm{m}}}$
1762	0.1972	0.1295	–	0.0353	–	0.0797	–
6460	0.0993	0.0660	0.9807	0.0184	0.9499	0.0430	0.8980
25101	0.0468	0.0322	0.9546	0.0093	0.8985	0.0210	0.9539
98357	0.0242	0.0160	1.0593	0.0046	1.0641	0.0105	1.0496
394038	0.0130	0.0080	1.1182	0.0023	1.1288	0.0052	1.1283
1668858	0.0061	0.0038	0.9694	0.0011	0.9579	0.0026	0.9084

dof	h_{Σ}	e_{λ}	r_{λ}	iteration
1762	0.0625	0.0735	–	8
6460	0.0312	0.0374	0.9734	8
25101	0.0156	0.0196	0.9268	8
98357	0.0078	0.0101	0.9510	8
394038	0.0039	0.0051	0.9718	8
1668858	0.0019	0.0025	1.0007	8

Table 3.3: EXAMPLE 2: Degree of Freedom, mesh sizes, errors, rates of convergence and number of iterations for the coupled problem with $\mu = 0.01$.

Chapter 4

An a posteriori error analysis for a nonisothermal fluid-membrane interaction

4.1 Introduction

In this chapter, we propose a residual-based a posteriori error estimator for a conforming discretization of the nonisothermal fluid-membrane model introduced in Chapter 3.

More precisely, we introduce a reliable and efficient a posteriori error estimator for the coupled system in Section 3.2. In this section, the primal formulation of the convection-diffusion equation for the temperature is coupled with the velocity-pressure formulation of the Navier–Stokes problem in the free-fluid region and the dual-mixed velocity-pressure formulation of the Darcy equation in the porous medium domain, which forces the introduction of a Lagrange multiplier representing the Darcy pressure at the interface.

The Galerkin scheme introduced in Section 3.3, considers the Bernardi–Raugel and Raviart–Thomas elements for the velocities, piecewise constant elements for pressures, continuous piecewise linear functions for temperatures, and continuous piecewise linear functions for the Lagrange multiplier on a partition of the interface.

Employing standard arguments such as global inf-sup conditions, suitable Helmholtz's decompositions and the local approximation properties of the Raviart–Thomas and Clément interpolation operators, we derive the a posteriori error estimator and prove its reliability. In turn, inverse inequalities, usual localization techniques of bubble functions and known results from previous works, are employed to prove the local efficiency of the proposed error estimator. Finally, we propose an adaptive algorithm based on the local and computable error indicators of the a posteriori error estimator.

The remainder of the chapter is organized as follows.

- In **Section 4.2**, we summarize the main results established in Chapter 3, which are necessary for the analysis that we will develop in this chapter.

- In **Section 4.3**, we derive a reliable and efficient residual-based a posteriori error estimator for the two dimensional case. Specifically, in Section 4.3.3, we employ auxiliary problems, suitable continuous inf-sup conditions, and local approximation properties of the Clément and Raviart–Thomas operators to prove the reliability of the estimator, whereas in Section 4.3.4, we apply inverse inequalities and the localization technique based on element-bubble and edge-bubble functions to prove the efficiency of the estimator.

- In **Section 4.4**, we extend the analysis developed in two dimensions to the three-dimensional case, proposing a reliable and efficient residual-based a posteriori error estimator for the 3D case.

4.2 The nonisotermal Navier–Stokes/Darcy coupled problem.

In this section, we briefly recall the modeling problem described in Chapter 2. In addition, we introduce both continuous and discrete variational formulations and summarize some of the results established in Chapter 3.

4.2.1 The model problem and its variational formulation

In order to describe the geometry of the problem, we let Ω_{f} and Ω_{m} be two bounded and simply connected open polygonal or polyhedral domains in $\mathbb{R}^d$, with $d = 2$ or $d = 3$, respectively, such that $\Omega_{\mathrm{f}} \cap \Omega_{\mathrm{m}} = \emptyset$ and $\Sigma := \partial\Omega_{\mathrm{f}} \cap \partial\Omega_{\mathrm{m}} \neq \emptyset$ is either the union of straight lines if $d = 2$ or a polygonal surface if $d = 3$, satisfying $\overline{\Sigma} \subsetneq \partial\Omega_{\mathrm{f}}$ and $\overline{\Sigma} \subsetneq \partial\Omega_{\mathrm{m}}$. Then, we let $\Gamma_{\mathrm{f}} := \partial\Omega_{\mathrm{f}} \setminus \overline{\Sigma}$, $\Gamma_{\mathrm{m}} := \partial\Omega_{\mathrm{m}} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_{\mathrm{f}} \cup \Omega_{\mathrm{m}} \cup \Sigma$ and Ω_{f} (and hence inward to Ω_{m} when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t} = \mathbf{t}_1$ when $d = 2$ and by $\mathbf{t}_1, \mathbf{t}_2$, when $d = 3$.

In the free fluid domain Ω_{f}, the phenomenon is described by the following

Navier–Stokes/Heat system:

$$\boldsymbol{\sigma}_{\mathrm{f}} = 2\mu\mathbf{e}(\mathbf{u}_{\mathrm{f}}) - p_{\mathrm{f}}\mathbf{I} \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{4.1a}$$

$$-\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{f}} + (\mathbf{u}_{\mathrm{f}}\cdot\nabla)\mathbf{u}_{\mathrm{f}} - \mathbf{g}_{\mathrm{f}}\theta_{\mathrm{f}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{4.1b}$$

$$\operatorname{div}\mathbf{u}_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{4.1c}$$

$$-\kappa_{\mathrm{f}}\Delta\theta_{\mathrm{f}} + \mathbf{u}_{\mathrm{f}}\cdot\nabla\theta_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{4.1d}$$

where $\mu > 0$ is the dynamic viscosity of the fluid, $\mathbf{u}_{\mathrm{f}}$ is the fluid velocity, p_{f} is the fluid pressure, $\boldsymbol{\sigma}_{\mathrm{f}}$ is the Cauchy stress tensor, $\mathbf{I}$ is the $d\times d$ identity matrix, θ_{f} is the fluid temperature, $\kappa_{\mathrm{f}} > 0$ is the fluid thermal conductivity, $\mathbf{g}_{\mathrm{f}} \in \mathbf{L}^2(\Omega_{\mathrm{f}})$ is the external force per unit mass, $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor, and $\mathbf{e}(\mathbf{u}_{\mathrm{f}})$ is the strain rate tensor given by $\mathbf{e}(\mathbf{u}_{\mathrm{f}}) := \frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{f}} + (\nabla\mathbf{u}_{\mathrm{f}})^t\right)$, where the superscript t denotes transposition.

In the porous membrane Ω_{m} the behavior of the fluid can be described by the following Darcy-Heat system,

$$\mathbf{K}^{-1}\mathbf{u}_{\mathrm{m}} + \nabla p_{\mathrm{m}} - \mathbf{g}_{\mathrm{m}}\theta_{\mathrm{m}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{4.2a}$$

$$\operatorname{div}\mathbf{u}_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{4.2b}$$

$$-\kappa_{\mathrm{m}}\Delta\theta_{\mathrm{m}} + \mathbf{u}_{\mathrm{m}}\cdot\nabla\theta_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{4.2c}$$

where $\mathbf{u}_{\mathrm{m}}$ represents the fluid velocity, p_{m} the fluid pressure, θ_{m} the fluid temperature, $\mathbf{g}_{\mathrm{m}} \in \mathbf{L}^3(\Omega_{\mathrm{m}})$ a given external force, $\kappa_{\mathrm{m}} > 0$ the thermal conductivity, and $\mathbf{K} \in [\mathrm{L}^\infty(\Omega_{\mathrm{m}})]^{d\times d}$ is a symmetric and uniformly positive definite tensor in Ω_{m} representing the intrinsic permeability $\boldsymbol{\kappa}$ of the membrane divided by the dynamic viscosity μ of the fluid.

The transmission conditions that couple the systems (4.1) and (4.2) on the interface Σ are given by

$$\begin{aligned}
&\theta_{\mathrm{f}} = \theta_{\mathrm{m}} && \text{on} \quad \Sigma,\\
&\kappa_{\mathrm{f}}\nabla\theta_{\mathrm{f}}\cdot\mathbf{n} = \kappa_{\mathrm{m}}\nabla\theta_{\mathrm{m}}\cdot\mathbf{n} && \text{on} \quad \Sigma,\\
&\mathbf{u}_{\mathrm{f}}\cdot\mathbf{n} = \mathbf{u}_{\mathrm{m}}\cdot\mathbf{n} && \text{on} \quad \Sigma,\\
&\boldsymbol{\sigma}_{\mathrm{f}}\mathbf{n} + \sum_{j=1}^{d-1}\omega_j(\mathbf{u}_{\mathrm{f}}\cdot\mathbf{t}_j)\mathbf{t}_j = -p_{\mathrm{m}}\mathbf{n} && \text{on} \quad \Sigma,
\end{aligned} \tag{4.3}$$

where $\omega_1, \ldots, \omega_{d-1}$ are positive constants depending on the intrinsic permeability $\boldsymbol{\kappa}$, the viscosity μ, and on the geometrical characteristics of the membrane. In particular, the fourth condition in (4.3) can be decomposed, at least formally, into its normal and tangential components as follows:

$$(\boldsymbol{\sigma}_{\rm f}\mathbf{n})\cdot\mathbf{n} = -p_{\rm m} \quad \text{and} \quad (\boldsymbol{\sigma}_{\rm f}\,\mathbf{n})\cdot\mathbf{t}_j \;=\; -\omega_j(\mathbf{u}_{\rm f}\cdot\mathbf{t}_j) \quad \text{on} \quad \Sigma, \quad j \in 1,\ldots,d-1. \tag{4.4}$$

The first equation in (4.4) corresponds to the balance of normal forces, whereas the second one is known as the Beavers–Joseph–Saffman law, which establishes that the slip velocity along Σ is proportional to the shear stress along Σ (assuming also, based on experimental evidence, that $\mathbf{u}_{\rm m}\cdot\mathbf{t}$ is negligible).

Finally, the Navier–Stokes/Darcy/heat system (4.1), (4.2) and (4.3) is complemented with suitable boundary conditions:

$$\begin{aligned} &\mathbf{u}_{\rm f} = \mathbf{0} \quad &&\text{on} \quad \Gamma_{\rm f}, \qquad &&\mathbf{u}_{\rm m}\cdot\mathbf{n} = 0 \quad &&\text{on} \quad \Gamma_{\rm m},\\ &\theta_{\rm f} = \theta_{\rm D}|_{\Gamma_{\rm f}} \quad &&\text{on} \quad \Gamma_{\rm f}, \qquad &&\theta_{\rm m} = \theta_{\rm D}|_{\Gamma_{\rm m}} \quad &&\text{on} \quad \Gamma_{\rm m}, \end{aligned} \tag{4.5}$$

where $\theta_{\rm D} \in \mathrm{W}^{3/4,4}(\Gamma)$ is a given function defined on $\Gamma := \Gamma_{\rm f}\cup\Gamma_{\rm m}$.

4.2.2 The variational formulation

To present a variational formulation for the coupled problem given by (4.1), (4.2), (4.3) and (4.5), let us first recall some notations and definitions employed in Chapter 3, Section 3.2.1. First, we define the spaces

$$\begin{aligned} &\mathbf{H}^1_{\Gamma_\star}(\Omega_\star) := \big\{\mathbf{v}_\star \in \mathbf{H}^1(\Omega_\star) : \quad \mathbf{v}_\star = \mathbf{0} \quad \text{on} \quad \Gamma_\star\big\} \quad \text{for } \star \in \{\mathrm{f},\mathrm{m}\},\\ &\mathbf{H}_{\Gamma_{\rm m}}(\mathrm{div};\Omega_{\rm m}) := \big\{\mathbf{v}_{\rm m} \in \mathbf{H}(\mathrm{div};\Omega_{\rm m}) : \quad \mathbf{v}_{\rm m}\cdot\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\rm m}\big\},\\ &\Psi_\infty := \big\{\psi \in \mathrm{H}^1(\Omega) \,:\, \psi|_{\Omega_{\rm m}} \in \mathrm{L}^\infty(\Omega_{\rm m})\big\}, \quad \text{and} \quad \Psi_{\infty,0} := \Psi_\infty \cap \mathrm{H}^1_0(\Omega), \end{aligned}$$

and let

$$p := p_{\rm f}\chi_{\rm f} + p_{\rm m}\chi_{\rm m} \quad \text{and} \quad \theta := \theta_{\rm f}\chi_{\rm f} + \theta_{\rm m}\chi_{\rm m},$$

where $\chi_\star$ being the characteristic function:

$$\chi_\star := \begin{cases} 1 & \text{in} \quad \Omega_\star,\\ 0 & \text{in} \quad \Omega\setminus\overline{\Omega}_\star, \end{cases}$$

for $\star \in \{\mathrm{f},\mathrm{m}\}$. Furthermore, we introduce the pressure trace of the porous medium on Σ as an additional unknown, that is

$$\lambda := p_{\rm m}|_\Sigma \in \mathrm{H}^{1/2}(\Sigma).$$

In addition, we group the unknowns and spaces as follows

$$\mathbf{u} := (\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}}) \in \mathbf{H} := \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}}),$$

$$(p,\lambda) \in \mathrm{Q} := \mathrm{L}^2_0(\Omega) \times \mathrm{H}^{1/2}(\Sigma),$$

where $\mathbf{H}$ and Q are endowed with the norms

$$\|\mathbf{v}\|_{\mathbf{H}} := \|\mathbf{v}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} + \|\mathbf{v}_{\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} \quad \text{and} \quad \|(q,\xi)\|_{\mathrm{Q}} := \|q\|_{0,\Omega} + \|\xi\|_{1/2,\Sigma},$$

respectively, and proceeding analogously to Section 3.2.1, we arrive at the following variational problem: Find $\mathbf{u} = (\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}}) \in \mathbf{H}$, $(p,\lambda) \in \mathrm{Q}$ and $\theta \in \mathrm{H}^1(\Omega)$, with $\theta|_{\Gamma} = \theta_{\mathrm{D}}$, such that:

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) + B(\mathbf{v},(p,\lambda)) - D(\theta,\mathbf{v}) &= 0, \\ B(\mathbf{u},(q,\xi)) &= 0, \\ A_{\mathrm{T}}(\theta,\psi) + O_{\mathrm{T}}(\mathbf{u};\theta,\psi) &= 0, \end{aligned} \tag{4.6}$$

for all $\mathbf{v} = (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{H}$, for all $(q,\xi) \in \mathrm{Q}$, and for all $\psi \in \Psi_{\infty,0}$. Above, the forms $A_{\mathrm{F}} : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$, $O_{\mathrm{F}} : \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \to \mathbb{R}$, $B : \mathbf{H} \times \mathrm{Q} \to \mathbb{R}$, $D : \mathrm{H}^1(\Omega) \times \mathbf{H} \to \mathbb{R}$, $A_{\mathrm{T}} : \mathrm{H}^1(\Omega) \times \Psi_{\infty,0} \to \mathbb{R}$, and $O_{\mathrm{T}} : \mathbf{H} \times \mathrm{H}^1(\Omega) \times \Psi_{\infty,0} \to \mathbb{R}$, are defined respectively, as

$$\begin{aligned} &A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) && := 2\mu(\mathbf{e}(\mathbf{u}_{\mathrm{f}}),\mathbf{e}(\mathbf{v}_{\mathrm{f}}))_{\Omega_{\mathrm{f}}} + \sum_{j=1}^{d-1} \omega_j \left\langle \mathbf{u}_{\mathrm{f}} \cdot \mathbf{t}_j, \mathbf{v}_{\mathrm{f}} \cdot \mathbf{t}_j \right\rangle_{\Sigma} + (\mathbf{K}^{-1}\mathbf{u}_{\mathrm{m}},\mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}}, \\ &A_{\mathrm{T}}(\theta,\psi) && := \kappa_{\mathrm{f}}(\nabla\theta,\nabla\psi)_{\Omega_{\mathrm{f}}} + \kappa_{\mathrm{m}}(\nabla\theta,\nabla\psi)_{\Omega_{\mathrm{m}}}, \\ &B(\mathbf{v},(q,\lambda)) && := -(q,\mathrm{div}\,\mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} - (q,\mathrm{div}\,\mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}} + \left\langle \mathbf{v}_{\mathrm{f}}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{m}}\cdot\mathbf{n}, \lambda \right\rangle_{\Sigma}, \\ &D(\theta,\mathbf{v}) && := (\theta\mathbf{g}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} + (\theta\mathbf{g}_{\mathrm{m}},\mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}}, \\ &O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) && := ((\mathbf{w}_{\mathrm{f}}\cdot\nabla)\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}}, \\ &O_{\mathrm{T}}(\mathbf{w};\theta,\psi) && := (\mathbf{w}_{\mathrm{f}}\cdot\nabla\theta,\psi)_{\Omega_{\mathrm{f}}} + (\mathbf{w}_{\mathrm{m}}\cdot\nabla\theta,\psi)_{\Omega_{\mathrm{m}}}. \end{aligned} \tag{4.7}$$

4.2.3 The continuous analysis

In this section, we recall some of the results presented in Chapter 3, Section 3.2.2, concerning the establishment of existence, uniqueness, and stability results for (4.6).

We begin by introducing a suitable lifting extension to handle the non-homogeneous Dirichlet boundary condition for the temperature, to achieve this, we denote by $\mathrm{E} : \mathrm{W}^{3/4,4}(\Gamma) \to \mathrm{W}^{1,4}(\Omega)$ the usual lifting operator (see for instance [47, Corollary B.53]), satisfying

$$\gamma_0(\mathrm{E}(\zeta)) = \zeta \quad \text{and} \quad \|\mathrm{E}(\zeta)\|_{1,4,\Omega} \leq c\|\zeta\|_{3/4,4,\Gamma} \qquad \forall\, \zeta \in \mathrm{W}^{3/4,4}(\Gamma),$$

where $\gamma_0 : \mathrm{W}^{1,4}(\Omega) \to \mathrm{W}^{3/4,4}(\Gamma)$ is the trace operator. In turn, we let $\delta > 0$, and similarly to [10, Lemma 2.8], define the function $\beta_\delta : \mathbb{R}^d \to \mathbb{R}$ given by

$$\beta_\delta(\mathbf{x}) := \begin{cases} 1 & \text{if } 0 \leq \mathrm{dist}(\mathbf{x},\Gamma) \leq \delta, \\ 2 - \delta^{-1}\mathrm{dist}(\mathbf{x},\Gamma) & \text{if } \delta \leq \mathrm{dist}(\mathbf{x},\Gamma) \leq 2\delta, \\ 0 & \text{if } \mathrm{dist}(\mathbf{x},\Gamma) \geq 2\delta, \end{cases}$$

where $\mathrm{dist}(\mathbf{x},\Gamma)$ denotes the distance from the point $\mathbf{x}$ to the boundary Γ. In this way, we introduce the following extension operator

$$\mathrm{E}_\delta := \beta_\delta \mathrm{E} : \mathrm{W}^{3/4,4}(\Gamma) \to \mathrm{W}^{1,4}(\Omega). \tag{4.8}$$

In the following lemma, we summarize some properties of this operator. For its proof, we refer the reader to Lemma 3.2.3.

Lemma 4.2.1. *For all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$, $\mathrm{E}_\delta(\zeta)$ lies in $\mathrm{L}^\infty(\Omega)$ and satisfies the estimate*

$$\|\mathrm{E}_\delta(\zeta)\|_{0,\infty,\Omega} \;\leq\; \|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta}. \tag{4.9}$$

In addition, there exist $C_{\mathrm{lift},1}$, $C_{\mathrm{lift},2}$, $\mathrm{C}_{\mathrm{lift},3} > 0$, such that

$$\|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} \;\leq\; C_{\mathrm{lift},1}\, \delta^{1/12}\, \|\zeta\|_{3/4,4,\Gamma} \qquad \forall \zeta \in \mathrm{W}^{3/4,4}(\Gamma), \tag{4.10a}$$

$$\|\mathrm{E}_\delta(\zeta)\|_{1,\Omega} \;\leq\; C_{\mathrm{lift},2}\, \delta^{-3/4}(1+\delta^2)^{1/2}\, \|\zeta\|_{3/4,4,\Gamma} \quad \forall \zeta \in \mathrm{W}^{3/4,4}(\Gamma), \tag{4.10b}$$

$$\|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega} \;\leq\; C_{\mathrm{lift},3}\, (1+\delta^{-2})^{1/2}\|\zeta\|_{3/4,4,\Gamma} \qquad \forall \zeta \in \mathrm{W}^{3/4,4}(\Gamma). \tag{4.10c}$$

The following theorem establishes the existence and stability of problem (4.6).

Theorem 4.2.2. *Let $\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$ with $\delta > 0$ satisfying*

$$c_1\alpha_\mathrm{F}^{-1}\alpha_\mathrm{T}^{-1}\gamma_\mathbf{g}\, \delta^{1/12}\, \|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \leq 1, \tag{4.11}$$

and assume that the lifting θ_1 satisfies

$$c_2\alpha_\mathrm{F}^{-1}\alpha_\mathrm{T}^{-1}\, \gamma_\mathbf{g}\, \|\theta_1\|_{0,\infty,\Omega} \;\leq\; 1. \tag{4.12}$$

In addition, assume that the datum θ_D satisfies

$$C_\mathbf{u}\, \gamma_\mathbf{g}\, \delta^{-3/4}(1+\delta^2)^{1/2}\, \|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \;\leq\; c_3\mu, \tag{4.13}$$

Then, there exists at least one solution $(\mathbf{u},(p,\lambda),\theta) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{H}^1(\Omega)$ *to* (4.6) *and satisfies*

$$\|\mathbf{u}\|_{\mathbf{H}} \leq C_{\mathbf{u}}\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega}, \qquad \|\theta_0\|_{1,\Omega} \leq C_\theta\|\theta_1\|_{1,\Omega}, \tag{4.14}$$

and

$$\|(p,\lambda)\|_{\mathrm{Q}} \leq c_4(C_{A_{\mathrm{F}}}C_{\mathbf{u}}\gamma_{\mathbf{g}} + C_{\mathbf{u}}^2\gamma_{\mathbf{g}}^2 + (C_\theta+1)\gamma_{\mathbf{g}})\|\theta_1\|_{1,\Omega}, \tag{4.15}$$

where $\theta_0 = \theta - \theta_1 \in \mathrm{H}_0^1(\Omega)$*. Above* $\gamma_{\mathbf{g}} := \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$, $\alpha_{\mathrm{F}} := \frac{1}{2}\min\{c_5\mu, C_{\mathbf{K}}\}$, $\alpha_{\mathrm{T}} := c_6\min\{\kappa_{\mathrm{f}},\kappa_{\mathrm{m}}\}$ *and* $C_{\mathbf{u}} := c_7\alpha_{\mathrm{F}}^{-1}\alpha_{\mathrm{T}}^{-1}(C_{A_{\mathrm{T}}} + \alpha_{\mathrm{T}})$ *are parameters,* $C_{A_{\mathrm{F}}}$ *and* $C_{A_{\mathrm{T}}}$ *are the continuity constants of the bilinear forms* A_{F} *and* A_{T}*, respectively, and* $c_1,\dots,c_7 > 0$ *are constants independent of the physical parameters.*

Proof. The existence of solution follows from Theorem 3.2.9 and Lemma 3.2.4, whereas the estimates (4.14) and (4.15) can be found in Theorem 3.2.9 and Corollary 3.2.10, respectively. We omit further details. □

We conclude this section by showing the uniqueness of solution to problem (4.6). The proof can be found in Theorem 3.2.11.

Theorem 4.2.3. *Assume that the hypotheses of Theorem 4.2.2 hold and let* $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{H}^1(\Omega)$ *be a solution of* (4.6)*. Assume further that* $\theta_0|_{\Omega_{\mathrm{m}}} \in \mathrm{W}^{1,4}(\Omega_{\mathrm{m}})$*, and*

$$\left[(M_1\gamma_{\mathbf{g}} + M_2)\,C_{\mathrm{lift},2}\delta^{-3/4}(1+\delta^2)^{1/2} + M_3 C_{\mathrm{lift},3}(1+\delta^{-2})^{1/2}\right]\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \\ + M_3\|\theta_0\|_{1,4,\Omega_{\mathrm{m}}} + M_4\gamma_{\mathbf{g}} < 1,$$

where $M_1 := c_8\alpha_{\mathrm{F}}^{-1}C_{\mathbf{u}}$, $M_2 := c_9\alpha_{\mathrm{T}}^{-1}\,(C_\theta+1)$, $M_3 := c_{10}\alpha_{\mathrm{T}}^{-1}$ *and* $M_4 := c_{11}\alpha_{\mathrm{F}}^{-1}$*, Then* $(\mathbf{u},(p,\lambda),\theta)$ *is unique. Above* $\gamma_{\mathbf{g}}$, α_{F}, α_{T}, $C_{\mathbf{u}}$ *and* C_θ *are the parameters defined in Theorem 4.2.2, and* $c_8,\dots,c_{11} > 0$ *are constants independent of the physical parameters.*

4.2.4 The conforming finite element method

Let $\mathcal{T}_h^{\mathrm{f}}$ and $\mathcal{T}_h^{\mathrm{m}}$ be the respective regular triangulations of the domains Ω_{f} and Ω_{m} formed by shape-regular triangles T in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$ of diameter h_T and denote by h_{f} and h_{m} their corresponding mesh sizes. Assume that they match on Σ so that $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{f}} \cup \mathcal{T}_h^{\mathrm{m}}$ is a triangulation of $\Omega := \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$. Hereafter $h := \max\{h_{\mathrm{f}}, h_{\mathrm{m}}\}$.

In what follows, we summarize some of the results presented in Chapter 3, Section 3.3. To do this, given an integer $l \geq 0$ and a subset S of $\mathbb{R}^d$, we denote by $\mathrm{P}_l(T)$ the space of polynomials defined in S of degree equal or less than l.

Moreover, for each $T \in \mathcal{T}_h^{\mathrm{f}}$, we denote by $\mathbf{BR}(T)$ the local Bernardi–Raugel space (see [11, 62])

$$\mathbf{BR}(T) := [\mathrm{P}_1(T)]^d \oplus \mathrm{span}\{\mathbf{p}_i \,:\, 1 \leq i \leq d+1\},$$

with $\mathbf{p}_i = \mathbf{n}_i \prod_{j=1, j\neq i}^{d+1} \eta_j$, where $\{\eta_1, \ldots, \eta_{d+1}\}$ are the baricentric coordinates of T, $\{\mathbf{n}_1, \ldots, \mathbf{n}_{d+1}\}$ are the unit outward normals to the opposite faces of the corresponding vertices of T. In turn, for each $T \in \mathcal{T}_h^{\mathrm{m}}$, we consider the local Raviart–Thomas space of the lowest order $\mathbf{RT}_0(T)$ (see [91])

$$\mathbf{RT}_0(T) := [\mathrm{P}_0(T)]^d \oplus \mathrm{P}_0(T)\mathbf{x},$$

where $\mathbf{x} := (x_1, \ldots, x_d)^t$ is a generic vector of $\mathbb{R}^d$.

Hence, we introduce the following finite element subspaces:

$$\begin{aligned}
&\mathbf{H}_h(\Omega_{\mathrm{f}}) := \{\mathbf{v}_{\mathrm{f}} \in \mathbf{H}^1(\Omega_{\mathrm{f}}) : \mathbf{v}_{\mathrm{f}}|_T \in \mathbf{BR}(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{f}}\},\\
&\mathbf{H}_h(\Omega_{\mathrm{m}}) := \{\mathbf{v}_{\mathrm{m}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{m}}) : \mathbf{v}_{\mathrm{m}}|_T \in \mathbf{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{m}}\},\\
&\mathrm{L}_h(\Omega) := \{q \in \mathrm{L}^2(\Omega) : q|_T \in \mathrm{P}_0(T) \quad \forall\, T \in \mathcal{T}_h\},\\
&\Psi_h := \{\psi \in \mathrm{H}^1(\Omega) : \psi|_T \in \mathrm{P}_1(T) \quad \forall\, T \in \mathcal{T}_h\},\\
&\mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) := \mathbf{H}_h(\Omega_{\mathrm{f}}) \cap \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}),\\
&\mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}}) := \mathbf{H}_h(\Omega_{\mathrm{m}}) \cap \mathrm{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}}),\\
&\mathrm{L}_{h,0}(\Omega) := \mathrm{L}_h(\Omega) \cap \mathrm{L}^2_0(\Omega),\\
&\Psi_{h,0} := \Psi_h \cap \mathrm{H}^1_0(\Omega).
\end{aligned} \tag{4.16}$$

In turn, to introduce the finite element subspace for $\mathrm{H}^{1/2}(\Sigma)$, we proceed differently for the 2D and 3D cases. For the two-dimensional case, we denote by Σ_h the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{f}}$ (or $\mathcal{T}_h^{\mathrm{m}}$) and assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. According to the above, we define the following finite element subspace for $\mathrm{H}^{1/2}(\Sigma)$

$$\Lambda_h(\Sigma) := \{\xi \in \mathrm{C}(\Sigma) : \xi|_e \in \mathrm{P}_1(e) \quad \forall\, e \in \Sigma_{2h}\}.$$

On the other hand, for the three-dimensional case, we again denote by Σ_h the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{f}}$ (or $\mathcal{T}_h^{\mathrm{m}}$), which is formed by triangles e of diameter h_e, and set $h_\Sigma := \max\{h_e : e \in \Sigma_h\}$. Then, we introduce an independent triangulation $\widehat{\Sigma}_h$ of Σ, formed by triangles $\hat{e}$ of diameter $h_{\hat{e}}$, and define the associated mesh-size $h_{\widehat{\Sigma}} := \max\{h_{\hat{e}} : \hat{e} \in \widehat{\Sigma}_h\}$. Therefore, the finite element subspace for $\mathrm{H}^{1/2}(\Sigma)$ is defined by

$$\Lambda_h(\Sigma) := \{\xi \in \mathrm{C}(\Sigma) : \xi|_{\hat{e}} \in \mathrm{P}_1(\hat{e}) \quad \forall\, \hat{e} \in \widehat{\Sigma}_h\}.$$

Now, we let $\mathrm{I}_h : \mathrm{C}(\overline{\Omega}) \to \Psi_h$ be the well-known Lagrange interpolation operator and recall that, under the assumption $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ and for a given $\delta > 0$, $\mathrm{E}_\delta(\theta_{\mathrm{D}})$ belongs to $\mathrm{W}^{1,4}(\Omega) \subseteq \mathrm{C}(\overline{\Omega})$. Then, for a fixed $\delta > 0$ (to be specified below), we define the following approximation to θ_{D}:

$$\theta^{\delta}_{\mathrm{D},h} = \mathrm{I}_h(\mathrm{E}_\delta(\theta_{\mathrm{D}}))|_\Gamma \in \{\psi_{\mathrm{D},h} \in \mathrm{C}(\Gamma) : \psi_{\mathrm{D},h}|_e \in \mathrm{P}_1(e) \text{ for all } e \in \mathcal{E}_h(\Gamma)\}, \tag{4.17}$$

where $\mathcal{E}_h(\Gamma)$ stands for the set of edges/faces on Γ. According to the above, denoting the global finite element spaces

$$\mathbf{H}_h := \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}}) \quad \text{and} \quad \mathrm{Q}_h := \mathrm{L}_{h,0}(\Omega) \times \Lambda_h(\Sigma),$$

the Galerkin scheme associated to problem (4.6) reads: Find $\mathbf{u}_h := (\mathbf{u}_{h,\mathrm{f}}, \mathbf{u}_{h,\mathrm{m}}) \in \mathbf{H}_h$, $(p_h, \lambda_h) \in \mathrm{Q}_h$ and $\theta_h \in \Psi_h$, such that $\theta_h|_\Gamma = \theta^{\delta}_{\mathrm{D},h}$, and

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u}_h, \mathbf{v}) + O^h_{\mathrm{F}}(\mathbf{u}_{h,\mathrm{f}}; \mathbf{u}_{h,\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) + B(\mathbf{v}, (p_h, \lambda_h)) - D(\theta_h, \mathbf{v}) &= 0, \\ B(\mathbf{u}_h, (q, \xi)) &= 0, \\ A_{\mathrm{T}}(\theta_h, \psi) + O^h_{\mathrm{T}}(\mathbf{u}_h; \theta_h, \psi) &= 0, \end{aligned} \tag{4.18}$$

for all $\mathbf{v} := (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{H}_h$, for all $(q, \xi) \in \mathrm{Q}_h$, and for all $\psi \in \Psi_{h,0}$. The forms A_{F}, A_{T}, B, and D are the same defined in Section 4.2.2, while O^h_{F} and O^h_{T} are the skew-symmetric convection forms (see [99]), defined respectively by

$$O^h_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}}; \mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) := O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}}; \mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) + \tfrac{1}{2}(\operatorname{div} \mathbf{w}_{\mathrm{f}}, \mathbf{u}_{\mathrm{f}}\, \mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}}, \tag{4.19}$$

for all $\mathbf{w}_{\mathrm{f}}, \mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$, and

$$O^h_{\mathrm{T}}(\mathbf{w}; \theta, \psi) := O_{\mathrm{T}}(\mathbf{w}; \theta, \psi) + \tfrac{1}{2}(\operatorname{div} \mathbf{w}_{\mathrm{f}}, \theta\, \psi)_{\Omega_{\mathrm{f}}}, \tag{4.20}$$

for all $\mathbf{w} = (\mathbf{w}_{\mathrm{f}}, \mathbf{w}_{\mathrm{m}}) \in \mathbf{H}_h$, for all $\theta, \psi \in \Psi_h$.

Now, analogously at the continuous case, we introduce a discrete extension operator. To this end, we note that since Ω is a polyhedral domain, Ω_δ is also a polyhedron that can be discretized by shaped-regular elements. Therefore, if $\mathcal{T}^{\delta}_h$ is a triangulation of Ω_δ, we assume that $\mathcal{T}^{\delta}_h \subseteq \mathcal{T}_h$. In this way, proceeding as in Section 3.3.2, given $\delta > 0$, we introduce the discrete extension operator $\mathrm{E}_{\delta,h} : \mathrm{W}^{3/4,4}(\Gamma) \to \Psi_h$, defined by $\mathrm{E}_{\delta,h} := \mathrm{I}_h \mathrm{E}_\delta$, where E_δ is the extension operator defined in (4.8) and I_h is the Lagrange interpolation operator. Then, it is clear from (4.17) that there holds

$$\theta^{\delta}_{\mathrm{D},h} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})|_\Gamma.$$

Additional properties of the operator $\mathrm{E}_{\delta,h}$ are summarized in the following lemma, the proof of which can be found in Lemma 3.3.4.

Lemma 4.2.4. *The following estimates hold:*

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,3,\Omega} \le \widehat{C}_{\mathrm{lift},1}\,\delta^{1/12}(h\delta^{-1}+h+1)\,\|\zeta\|_{3/4,4,\Gamma}, \tag{4.21a}$$
$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{1,\Omega} \le \widehat{C}_{\mathrm{lift},2}\,\delta^{1/4}(2+\delta^{-1})\,\|\zeta\|_{3/4,4,\Gamma}, \tag{4.21b}$$
$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,\infty,\Omega} \le (d+1)\|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta}, \tag{4.21c}$$

for all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$*, where* $\widehat{C}_{\mathrm{lift},1}$*,* $\widehat{C}_{\mathrm{lift},2} > 0$ *are constants independent of* h *and* δ*.*

We also recall the discrete kernel of the bilinear form B given by

$$\mathbf{V}_h := \left\{ \begin{array}{ll} \mathbf{v} = (\mathbf{v}_\mathrm{f}, \mathbf{v}_\mathrm{m}) \in \mathbf{H}_h : & (q, \operatorname{div}\mathbf{v}_\mathrm{f})_{\Omega_\mathrm{f}} = 0 \quad \forall\, q \in \mathrm{L}_h(\Omega_\mathrm{f}), \\ & \operatorname{div}\mathbf{v}_\mathrm{m} = 0 \quad \text{in} \quad \Omega_\mathrm{m} \quad \text{and} \\ & \langle \mathbf{v}_\mathrm{f}\cdot\mathbf{n} - \mathbf{v}_\mathrm{m}\cdot\mathbf{n}, \xi\rangle_\Sigma = 0 \quad \forall\, \xi \in \Lambda_h(\Sigma) \end{array} \right\},$$

and where $\mathrm{L}_h(\Omega_\mathrm{f})$ is the set of functions of $\mathrm{L}_h(\Omega)$ restricted to Ω_f.

The next lemma establish the discrete inf-sup condition. The proof can be found in Lemma 3.3.3.

Lemma 4.2.5. *Given* $h = \max\{h_\mathrm{f}, h_\mathrm{m}\} > 0$*, let* $\mathbf{H}_h$ *and* Q_h *be the discrete space introduced in* (4.16) *and, in particular for the three-dimensional case, assume that* Σ_h *is quasi-uniform and that the meshes* Σ_h *and* $\Sigma_{\widehat{h}}$ *satisfy*

$$h_\Sigma \le C_0\widehat{h}_\Sigma, \quad \text{with} \quad C_0 \in (0,1).$$

Then, there exist $h_0 > 0$ *and* $\widehat{\beta} > 0$*, independent of* h*, such that for any* $h_\mathrm{m} \le h_0$*, there holds:*

$$\sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_\mathbf{H}} \ge \widehat{\beta}\,\|(q,\xi)\|_\mathrm{Q} \quad \forall\,(q,\xi)\in\mathrm{Q}_h.$$

The following theorem establishes the well-posedness and convergence of the discrete scheme (4.18).

Theorem 4.2.6. *Given* $h = \max\{h_\mathrm{f}, h_\mathrm{m}\} > 0$*, and in particular for the three-dimensional case, we assume that the meshes* Σ_h *and* $\widehat{\Sigma}_h$ *satisfy* $h_\Sigma \le C_0 h_{\widehat{\Sigma}}$ *with* $C_0 \in (0,1)$*, and that* Σ_h *is quasi-uniform. In addition, let* $h_0 > 0$ *be the fixed constant that allows to derive the inf-sup condition given in Lemma 4.2.5, and assume that* $h \le \min\{h_0, \delta\}$*, with* $\delta > 0$ *satisfying*

$$\widehat{c}_1\alpha_\mathrm{F}^{-1}\alpha_\mathrm{T}^{-1}\gamma_\mathbf{g}\delta^{1/12}(2+\delta)\|\theta_\mathrm{D}\|_{3/4,4,\Gamma} < 1, \tag{4.22}$$

where $\gamma_{\mathbf{g}} := \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$. *Let* $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ *be such that*

$$\widehat{c}_2 \alpha_{\mathrm{F}}^{-1} \alpha_{\mathrm{T}}^{-1} \gamma_{\mathbf{g}} \|\theta_{h,1}\|_{0,\infty,\Omega} < 1, \tag{4.23}$$

and assume further that $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ *satisfies*

$$C_{\mathbf{u}} \gamma_{\mathbf{g}} \delta^{1/4}(2 + \delta^{-1}) \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \leq \widehat{c}_3 \mu. \tag{4.24}$$

Then, there exists at least one $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ *solution to* (4.18). *Moreover, the solution satisfies*

$$\|\mathbf{u}_h\|_{\mathbf{H}} \leq C_{\mathbf{u}} \gamma_{\mathbf{g}} \|\theta_{h,1}\|_{1,\Omega}, \qquad \|\theta_{h,0}\|_{1,\Omega} \leq C_\theta \, \|\theta_{h,1}\|_{1,\Omega}, \tag{4.25}$$

and

$$\|(p_h, \lambda_h)\|_{\mathrm{Q}} \leq \widehat{c}_4 (C_{A_{\mathrm{F}}} C_{\mathbf{u}} \gamma_{\mathbf{g}} + C_{\mathbf{u}}^2 \gamma_{\mathbf{g}}^2 + (C_\theta + 1)\gamma_{\mathbf{g}}) \|\theta_{h,1}\|_{1,\Omega}, \tag{4.26}$$

where $\theta_{h,0} = \theta_h - \theta_{h,1} \in \Psi_{h,0}$.

In addition, assuming that $\mathbf{u}_{\mathrm{f}} \in \mathbf{H}^2(\Omega_{\mathrm{f}})$, $\mathbf{u}_{\mathrm{m}} \in \mathbf{H}^1(\Omega_{\mathrm{m}})$, $\mathrm{div}\,\mathbf{u}_{\mathrm{m}} \in \mathrm{H}^1(\Omega_{\mathrm{m}})$, $p \in \mathrm{H}^1(\Omega)$, $\lambda \in \mathrm{H}^{3/2}(\Sigma)$, $\theta|_{\Omega_{\mathrm{f}}} \in \mathrm{H}^2(\Omega_{\mathrm{f}})$ *and* $\theta|_{\Omega_{\mathrm{m}}} \in \mathrm{W}^{2,4}(\Omega_{\mathrm{m}})$, *and the following estimate holds*

$$\widehat{c}_5 \gamma_{\mathbf{g}} \delta^{-3/4} (1 + \delta^2)^{1/2} \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + \widehat{c}_6 \gamma_{\mathbf{g}} \|\theta\|_{1,4,\Omega_{\mathrm{m}}} \leq 1,$$

there exists $C_{rate} > 0$, *independent of the mesh-size and the solution, such that*

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}} + \|(p - p_h, \lambda - \lambda_h)\|_{\mathrm{Q}} + \|\theta - \theta_h\|_{1,\Omega} \leq C_{rate} h \big(\|\mathbf{u}_{\mathrm{f}}\|_{2,\Omega_{\mathrm{f}}} + \|\mathbf{u}_{\mathrm{m}}\|_{1,\Omega_{\mathrm{m}}} \\ + \|\mathrm{div}\,\mathbf{u}_{\mathrm{m}}\|_{1,\Omega_{\mathrm{m}}} + \|p\|_{1,\Omega} + \|\lambda\|_{3/2,\Sigma} + \|\theta\|_{2,\Omega_{\mathrm{f}}} + \|\theta\|_{2,4,\Omega_{\mathrm{m}}}\big).\end{aligned}$$

Above $\widehat{c}_1, \ldots, \widehat{c}_6 > 0$ *are constants independent of the mesh-size, data and physical parameters.*

Proof. We refer the reader to Theorems 3.3.6 and 3.4.4 for details. □

4.3 A posteriori error analysis for the two-dimensional case

In this section, we derive a reliable and efficient residual-based a posteriori error estimator for the two-dimensional version of the numerical scheme (4.18), under suitable hypotheses. Subsequently, in Section 4.4, we extend the analysis to the three-dimensional case. To that end, throughout this section, we assume that the hypotheses of Theorem 4.2.3 hold and let $(\mathbf{u}, (p, \lambda), \theta) = ((\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}}), (p, \lambda), \theta_0 + \theta_1) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{H}^1(\Omega)$ be the unique solution of (4.6), where $\theta_1 = \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega)$ and

$\theta_0 \in \mathrm{H}^1_0(\Omega)$. Additionally, we assume that the hypotheses of Theorem 4.2.6 hold and let $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) = ((\mathbf{u}_{h,\mathrm{f}}, \mathbf{u}_{h,\mathrm{m}}), (p_h, \lambda_h), \theta_{h,0} + \theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ be a solution of (4.18), where $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_\mathrm{D}) \in \Psi_h$ and $\theta_{h,0} \in \Psi_{h,0}$.

We also assume, for the sake of simplicity, that the Dirichlet datum θ_D is a piecewise linear function on Γ, that is

$$\theta_\mathrm{D} \in \mathrm{P}_1(\Gamma_h) := \{\psi_{\mathrm{D},h} \in \mathrm{C}(\Gamma) : \psi_{\mathrm{D},h}|_e \in \mathrm{P}_1(e) \text{ for all } e \in \mathcal{E}_h(\Gamma)\}.$$

If this is not the case, high-order terms would appear in the reliability estimate.

4.3.1 Preliminaries

Given $T \in \mathcal{T}_h = \mathcal{T}_h^\mathrm{f} \cup \mathcal{T}_h^\mathrm{m}$, we let $\mathcal{E}(T)$ be the set of edges of T, and denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h$, subdivided as follows:

$$\mathcal{E}_h := \mathcal{E}_h(\Gamma_\mathrm{f}) \cup \mathcal{E}_h(\Gamma_\mathrm{m}) \cup \mathcal{E}_h(\Omega_\mathrm{f}) \cup \mathcal{E}_h(\Omega_\mathrm{m}) \cup \mathcal{E}_h(\Sigma),$$

where $\mathcal{E}_h(\Gamma_\star) := \{e \in \mathcal{E}_h : e \subseteq \Gamma_\star\}$, $\mathcal{E}_h(\Omega_\star) := \{e \in \mathcal{E}_h : e \subseteq \Omega_\star\}$ for each $\star \in \{\mathrm{f}, \mathrm{m}\}$, and $\mathcal{E}_h(\Sigma) := \{e \in \mathcal{E}_h : e \subseteq \Sigma\}$. Analogously, we let $\mathcal{E}_{2h}(\Sigma)$ be the set of edges of the auxiliary partition Σ_{2h} of Σ introduced in Section 4.2.4. In what follows, h_e stands for the diameter of a given edge $e \in \mathcal{E}_h \cup \mathcal{E}_{2h}(\Sigma)$.

Now, given $T \in \mathcal{T}_h^\mathrm{f} \cup \mathcal{T}_h^\mathrm{m}$, we let $q \in \mathrm{L}^2(\Omega)$ be such that $q|_T \in \mathrm{C}(T)$ and for each $e \in \mathcal{E}_h$, we denote by $[\![q]\!]$ the jump of q across e, that is $[\![q]\!] := (q|_T)|_e - (q|_{T'})|_e$, where T and T' are the triangles of $\mathcal{T}_h^\mathrm{f} \cup \mathcal{T}_h^\mathrm{m}$ having e as an edge. We also denote the unit normal vector on each of edge $e \in \mathcal{E}_h$ by $\mathbf{n}_e = (n_1, n_2)^t$ (its particular orientation is not relevant), and define the tangential vector $\mathbf{t}_e$ along e by $\mathbf{t}_e = (-n_2, n_1)^t$. Then, given $\mathbf{v} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{v}|_T \in [\mathrm{C}(T)]^2$, for each $T \in \mathcal{T}_h$, we let $[\![\mathbf{v}\cdot\mathbf{t}_e]\!]$ be the tangential jump of $\mathbf{v}$ across e, that is $[\![\mathbf{v}\cdot\mathbf{t}_e]\!] := \{(\mathbf{v}|_T)|_e - (\mathbf{v}|_{T'})|_e\}\cdot\mathbf{t}_e$. From now on, when no confusion arises, we simply write $\mathbf{n}$ and $\mathbf{t}$ instead $\mathbf{n}_e$ and $\mathbf{t}_e$, respectively.

Finally, for sufficiently smooth scalar and vector fields q and $\mathbf{v} := (v_1, v_2)$, respectively, we let

$$\mathrm{curl}\,(q) := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right)^t \quad \text{and} \quad \mathrm{rot}\,(\mathbf{v}) := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

Let us now recall the main properties of the Clément operator (cf. [31]) onto the space of continuous piecewise linear functions and the lowest order Raviart–Thomas interpolator (see eg. [18]).

For $\star \in \{\mathrm{f}, \mathrm{m}\}$, we consider the discrete space

$$\mathrm{X}_{h,\star} := \{v \in \mathrm{C}(\overline{\Omega}_\star) : v|_T \in \mathrm{P}_1(T) \quad \forall\, T \in \mathcal{T}_h^\star\} \qquad \forall \star \in \{\mathrm{f}, \mathrm{m}\}.$$

and denote by $\mathcal{C}_{h,\star} : \mathrm{H}^1(\Omega_\star) \to \mathrm{X}_{h,\star}$, $\star \in \{\mathrm{f}, \mathrm{m}\}$, the Clément interpolator operator.

A vector version of $\mathcal{C}_{h,\star}$, say $\boldsymbol{\mathcal{C}}_{h,\star} : \mathbf{H}^1(\Omega_\star) \to \mathbf{X}_{h,\star}$, which is defined componentwise by $\mathcal{C}_{h,\star}$, will also be required.

The local approximation properties of $\mathcal{C}_{h,\star}$ (and hence of $\boldsymbol{\mathcal{C}}_{h,\star}$) are established in the following lemma. For the respective proofs we refer the reader to [31].

Lemma 4.3.1. *There exist $\tilde{c}_1$, $\tilde{c}_2 > 0$, independent of the mesh-size, such that for all $v \in \mathrm{H}^1(\Omega_\star)$ there hold*

$$\|v - \mathcal{C}_{h,\star}(v)\|_{0,T} \leq \tilde{c}_1 h_T \|v\|_{1,\Delta_\star(T)} \qquad \forall\, T \in \mathcal{T}_h^\star,$$

and

$$\|v - \mathcal{C}_{h,\star}(v)\|_{0,e} \leq \tilde{c}_2 h_e^{1/2} \|v\|_{1,\Delta_\star(e)} \qquad \forall\, e \in \mathcal{E}_h,$$

where $\Delta_\star(T) := \cup\{T' \in \mathcal{T}_h^\star : T' \cap T \neq \emptyset\}$ and $\Delta_\star(e) := \cup\{T' \in \mathcal{T}_h^\star : T' \cap e \neq \emptyset\}$ and $\star \in \{\mathrm{f}, \mathrm{m}\}$.

Now we let $\Pi^{RT} : \mathbf{H}^1(\Omega_\mathrm{m}) \to \mathbf{H}_h(\Omega_\mathrm{m})$ be the lowest order Raviart–Thomas interpolation operator, which is characterized by the identity

$$(\Pi^{RT}(\mathbf{v}_\mathrm{m}) \cdot \mathbf{n}, 1)_e = (\mathbf{v}_\mathrm{m} \cdot \mathbf{n}, 1)_e \quad \forall\, \text{edge } e \text{ of } \mathcal{T}_h^\mathrm{m}, \quad \forall\, \mathbf{v}_\mathrm{m} \in \mathbf{H}^1(\Omega_\mathrm{m}). \tag{4.27}$$

It is well-known that (4.27) implies the following identity

$$\operatorname{div}(\Pi^{RT}(\mathbf{v}_\mathrm{m})) = \mathcal{P}_h(\operatorname{div}(\mathbf{v}_\mathrm{m})), \tag{4.28}$$

where $\mathcal{P}_h$ is the $\mathrm{L}^2(\Omega_\mathrm{m})$-orthogonal projection onto the piecewise constant functions on Ω_m.

The local approximation properties of Π^{RT} are established in the following lemma. For the corresponding proofs we refer the reader to [18].

Lemma 4.3.2. *There exist $\tilde{c}_3$, $\tilde{c}_4 > 0$, independent of the mesh-size, such that for all $\mathbf{v}_\mathrm{m} \in \mathbf{H}^1(\Omega_\mathrm{m})$, there hold*

$$\|\mathbf{v}_\mathrm{m} - \Pi^{RT}(\mathbf{v}_\mathrm{m})\|_{0,T} \leq \tilde{c}_3 h_T \|\mathbf{v}_\mathrm{m}\|_{1,T} \qquad \forall\, T \in \mathcal{T}_h^\mathrm{m},$$

and

$$\|\mathbf{v}_\mathrm{m} \cdot \mathbf{n} - \Pi^{RT}(\mathbf{v}_\mathrm{m}) \cdot \mathbf{n}\|_{0,e} \leq \tilde{c}_4 h_e^{1/2} \|\mathbf{v}_\mathrm{m}\|_{1,T_e} \qquad \forall\, \text{edge } e \text{ of } \mathcal{T}_h^\mathrm{m},$$

where T_e is a triangle of $\mathcal{T}_h^\mathrm{m}$ containing e on its boundary.

Moreover, for the forthcoming analysis we will also utilize a stable Helmholtz decomposition for $\mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m})$, which can be found in [26, Lemma 4.4] (see also [53, Theorem 3.1] for the three-dimensional case).

Lemma 4.3.3. *Assume that there exists a convex domain B such that $\Omega_\mathrm{m} \subseteq B$ and $\Gamma_\mathrm{m} \subseteq B$. Then, for each $\mathbf{v}_\mathrm{m} \in \mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m})$, there exist $\mathbf{z} \in \mathbf{H}^1(\Omega_\mathrm{m})$ and $\zeta \in \mathrm{H}^1_{\Gamma_\mathrm{m}}(\Omega_\mathrm{m})$, such that*

$$\mathbf{v}_\mathrm{m} = \mathbf{z} + \operatorname{curl}(\zeta) \quad \textit{in} \quad \Omega_\mathrm{m} \quad \textit{and} \quad \|\mathbf{z}\|_{1,\Omega_\mathrm{m}} + \|\zeta\|_{1,\Omega_\mathrm{m}} \leq \tilde{c}_5 \|\mathbf{v}_\mathrm{m}\|_{\mathrm{div};\Omega_\mathrm{m}}, \tag{4.29}$$

where $\tilde{c}_5$ is a positive constant independent of all the foregoing variables.

4.3.2 The residual-based estimator

We propose the following global a posteriori error estimator for the numerical scheme (4.18):

$$\boldsymbol{\eta} := \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{f}}} \eta_{T,\mathrm{f}}^2 + \sum_{T\in\mathcal{T}_h^{\mathrm{m}}} \eta_{T,\mathrm{m}}^2 + \sum_{T\in\mathcal{T}_h^{\Sigma}} \eta_{T,\Sigma}^2 \right\}^{1/2}, \tag{4.30}$$

where $\mathcal{T}_h^{\Sigma}$ is the set of triangles of $\mathcal{T}_h$ with edges on the interface Σ and the local error indicators $\eta_{T,\mathrm{f}}$ and $\eta_{T,\mathrm{m}}$ and $\eta_{T,\Sigma}$ are given by

$$\begin{aligned} \eta_{T,\mathrm{f}}^2 := \quad & \|\mathbf{div}\,\mathbf{u}_{h,\mathrm{f}}\|_{0,T}^2 + h_T^2 \|\mathbf{div}\,\boldsymbol{\sigma}_{h,\mathrm{f}} - (\mathbf{u}_{h,\mathrm{f}}\cdot\nabla)\mathbf{u}_{h,\mathrm{f}} - \tfrac{1}{2}\mathbf{u}_{h,\mathrm{f}}\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}} + \theta_h \mathbf{g}_{\mathrm{f}}\|_{0,T}^2 \\ & + h_T^2 \|\kappa_{\mathrm{f}}\Delta\theta_h - \mathbf{u}_{h,\mathrm{f}}\cdot\nabla\theta_h - \tfrac{1}{2}\theta_h \mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{f}})} h_e \|[\![\boldsymbol{\sigma}_{h,\mathrm{f}}\mathbf{n}]\!]\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{f}})} h_e \|[\![\kappa_{\mathrm{f}}\nabla\theta_h\cdot\mathbf{n}]\!]\|_{0,e}^2, \end{aligned} \tag{4.31}$$

$$\begin{aligned} \eta_{T,\mathrm{m}}^2 := \quad & h_T^2 \|\mathrm{rot}\,(\theta_h\mathbf{g}_{\mathrm{m}} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}})\|_{0,T}^2 + h_T^2\|\theta_h\mathbf{g}_{\mathrm{m}} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}\|_{0,T}^2 \\ & + h_T^2 \|\kappa_{\mathrm{m}}\Delta\theta_h - \mathbf{u}_{h,\mathrm{m}}\cdot\nabla\theta_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{m}})} h_e \|[\![(\theta_h\mathbf{g}_{\mathrm{m}} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}})\cdot\mathbf{t}]\!]\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{m}})} h_e \|[\![\kappa_{\mathrm{m}}\nabla\theta_h\cdot\mathbf{n}]\!]\|_{0,e}^2, \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} \eta_{T,\Sigma}^2 \quad := & \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \|\kappa_{\mathrm{f}}\nabla\theta_{h,\mathrm{f}}\cdot\mathbf{n} - \kappa_{\mathrm{m}}\nabla\theta_{h,\mathrm{m}}\cdot\mathbf{n}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\|\lambda_h - p_h\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \|\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n} - \mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n}\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \|\theta_h\mathbf{g}_{\mathrm{m}}\cdot\mathbf{t} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{t} - \tfrac{d\lambda_h}{d\mathbf{t}}\|_{0,e}^2 \\ & + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e \|\boldsymbol{\sigma}_{h,\mathrm{f}}\mathbf{n} + \omega_1(\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{t})\mathbf{t} + \lambda_h\mathbf{n}\|_{0,e}^2, \end{aligned} \tag{4.33}$$

with $\theta_{h,\star} = \theta_h|_{\Omega_\star}$, for $\star \in \{\mathrm{f},\mathrm{m}\}$ and

$$\boldsymbol{\sigma}_{h,\mathrm{f}} := 2\mu\mathbf{e}(\mathbf{u}_{h,\mathrm{f}}) - p_h\mathbf{I} \qquad \text{on each } T\in\mathcal{T}_h^{\mathrm{f}}. \tag{4.34}$$

The residual character of each term on the right-hand sides of (4.31), (4.32) and (4.33) is quite clear. In addition, the derivative $\frac{d\lambda_h}{d\mathbf{t}}$ has to be understood as tangential derivatives in the direction imposed by the tangential vector field $\mathbf{t}$ on Σ.

In what follows we establish, under suitable assumptions, the existence of positive constants C_{rel} and C_{eff}, independent of the mesh-size and the continuous and discrete solutions, such that

$$C_{eff}\boldsymbol{\eta} + \text{h.o.t.} \leq \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}} + \|(p - p_h, \lambda - \lambda_h)\|_{\mathrm{Q}} + \|\theta - \theta_h\|_{1,\Omega} \leq C_{rel}\boldsymbol{\eta}, \quad (4.35)$$

where h.o.t. denotes higher order terms. The upper and lower bounds in (4.35), which are known as the reliability and efficiency of $\boldsymbol{\eta}$, are derived below in Sections 4.3.3 and 4.3.4. We begin with the reliability estimate.

4.3.3 Reliability

The derivation of the reliability estimate is organized into several parts, presented in the following subsections. We begin by deriving preliminary estimates for the errors $\|(\mathbf{u} - \mathbf{u}_h, (p - p_h, \lambda - \lambda_h))\|_{\mathbf{H}\times\mathrm{Q}}$ and $\|\theta - \theta_h\|_{1,\Omega}$ in term of residuals.

Preliminary estimates

We begin the derivation of the reliability estimate by noticing that from assumption (4.13), the estimate for $\mathbf{u} = (\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}})$ in (4.14), (4.10b) and the trace inequality, and proceeding similarly to the proof of Lemma 3.2.6, we can deduce that $\|\mathbf{u}_{\mathrm{f}} \cdot \mathbf{n}\|_{0,\Sigma}$ satisfies the hypothesis of Lemma 3.2.1, thus $A_{\mathrm{F}}(\cdot,\cdot) + O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\cdot,\cdot)$ is elliptic on the kernel of the bilinear form B, denoted by $\mathbf{V}$, that is

$$A_{\mathrm{F}}(\mathbf{v},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}};\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) \geq \alpha_{\mathrm{F}}\|\mathbf{v}\|_{\mathbf{H}}^2 \qquad \forall\,\mathbf{v} \in \mathbf{V},$$

with $\alpha_{\mathrm{F}} > 0$, where $\mathbf{V}$ can be characterized as follows

$$\mathbf{V} := \left\{ \begin{array}{rl} \mathbf{v} = (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{H}: & \operatorname{div}\mathbf{v}_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \quad \operatorname{div}\mathbf{v}_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}} \\ & \text{and} \quad \mathbf{v}_{\mathrm{f}}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{m}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma \end{array} \right\}.$$

Then, recalling from [45, Lemma 1] that B satisfies the inf-sup condition:

$$\sup_{\mathbf{v}\in\mathbf{H}\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta\|(q,\xi)\|_{\mathrm{Q}} \qquad \forall\,(q,\xi) \in \mathrm{Q},$$

from [47, Proposition 2.36] we deduce that there exists $\gamma > 0$, which depends on α_{F} and β, such that the following global inf-sup estimate holds

$$
\begin{aligned}
&\gamma\|(\mathbf{w},(r,\varsigma))\|_{\mathbf{H}\times\mathrm{Q}}\\
&\le \sup_{(\mathbf{v},(q,\xi))\in\mathbf{H}\times\mathrm{Q}\setminus\{\mathbf{0}\}} \frac{|A_{\mathrm{F}}(\mathbf{w},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{w}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) + B(\mathbf{v},(r,\varsigma)) + B(\mathbf{w},(q,\xi))|}{\|(\mathbf{v},(q,\xi))\|_{\mathbf{H}\times\mathrm{Q}}},
\end{aligned}
\tag{4.36}
$$

for all $(\mathbf{w},(r,\varsigma)) \in \mathbf{H}\times\mathrm{Q}$. This estimate will be employed to obtain the following preliminary result.

Lemma 4.3.4. *Assume that*

$$
C_{O_{\mathrm{F}}} C_{\mathbf{u}} \widehat{C}_{\mathrm{lift},2}\delta^{1/4}(2+\delta^{-1})(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \le \frac{\gamma}{2}. \tag{4.37}
$$

The following estimate holds

$$
\|(\mathbf{u}-\mathbf{u}_h,(p-p_h,\lambda-\lambda_h))\|_{\mathbf{H}\times\mathrm{Q}} \le 2\gamma^{-1}\big(\|R^{\mathrm{F}}\| + C_D(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta-\theta_h\|_{1,\Omega}\big), \tag{4.38}
$$

where $R^{\mathrm{F}} : \mathbf{H}\times\mathrm{Q} \to \mathbb{R}$ *is the linear and bounded functional defined by*

$$
\begin{aligned}
R^{\mathrm{F}}(\mathbf{v},(q,\xi)) := \; & -A_{\mathrm{F}}(\mathbf{u}_h,\mathbf{v}) - O_{\mathrm{F}}^h(\mathbf{u}_{h,\mathrm{f}};\mathbf{u}_{h,\mathrm{f}},\mathbf{v}_{\mathrm{f}})\\
& -B(\mathbf{v},(p_h,\lambda_h)) - B(\mathbf{u}_h,(q,\xi)) + D(\theta_h,\mathbf{v}),
\end{aligned}
\tag{4.39}
$$

for all $(\mathbf{v},(q,\xi)) \in \mathbf{H}\times\mathrm{Q}$. *Above,* $\widehat{C}_{\mathrm{lift},2}$ *and* γ *are positive constants satisfying* (4.21b) *and* (4.36)*, respectively, while* $C_{O_{\mathrm{F}}} > 0$ *and* $C_D > 0$ *are the constants satisfying* (4.40)*.*

Proof. Taking $(\mathbf{w},(r,\varsigma)) = (\mathbf{u}-\mathbf{u}_h,(p-p_h,\lambda-\lambda_h))$ in (4.36), using the first and second equations of (4.6), employing the fact that $O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) = O_{\mathrm{F}}^h(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}})$ for all $\mathbf{u}_{\mathrm{f}}$, $\mathbf{v}_{\mathrm{f}} \in \mathbf{H}$, and adding and subtracting suitable terms, we first find that

$$
\begin{aligned}
&\gamma\|(\mathbf{u}-\mathbf{u}_h,(p-p_h,\lambda-\lambda_h))\|_{\mathbf{H}\times\mathrm{Q}}\\
&\qquad \le \sup_{(\mathbf{v},(q,\xi))\in\mathbf{H}\times\mathrm{Q}\setminus\{\mathbf{0}\}} \frac{|R^{\mathrm{F}}(\mathbf{v},(q,\xi))|}{\|(\mathbf{v},(q,\xi))\|_{\mathbf{H}\times\mathrm{Q}}}\\
&\qquad + \sup_{(\mathbf{v},(q,\xi))\in\mathbf{H}\times\mathrm{Q}\setminus\{\mathbf{0}\}} \frac{|-O_{\mathrm{F}}^h(\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}};\mathbf{u}_{h,\mathrm{f}},\mathbf{v}_{\mathrm{f}}) + D(\theta-\theta_h,\mathbf{v})|}{\|(\mathbf{v},(q,\xi))\|_{\mathbf{H}\times\mathrm{Q}}},
\end{aligned}
$$

where $R^{\mathrm{F}} \in [\mathbf{H}\times\mathrm{Q}]'$ is given by (4.39). Hence, using the estimates

$$
\begin{aligned}
|O_{\mathrm{F}}^h(\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}};\mathbf{u}_{h,\mathrm{f}},\mathbf{v}_{\mathrm{f}})| &\le C_{O_{\mathrm{F}}}\|\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}\|\mathbf{v}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}},\\
|D(\theta-\theta_h,\mathbf{v})| &\le C_D(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta-\theta_h\|_{1,\Omega}\|\mathbf{v}\|_{\mathbf{H}},
\end{aligned}
\tag{4.40}
$$

we deduce that

$$\begin{aligned}\gamma\|(\mathbf{u}-\mathbf{u}_h,(p-p_h,\lambda-\lambda_h))\|_{\mathbf{H}\times\mathrm{Q}} \\ \leq \|R^{\mathrm{F}}\| + C_{O_{\mathrm{F}}}\|\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} + C_D(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta-\theta_h\|_{1,\Omega}.\end{aligned}$$

In this way, from the latter, and from (4.21b) and (4.25), it readily follows that

$$\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \leq \|\mathbf{u}_h\|_{\mathbf{H}} \leq C_{\mathbf{u}}\widehat{C}_{\mathrm{lift},2}\delta^{1/4}(2+\delta^{-1})(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma},$$

which combined with (4.37) and the fact that $\|\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \leq \|(\mathbf{u}-\mathbf{u}_h,(p-p_h,\lambda-\lambda_h))\|_{\mathbf{H}\times\mathrm{Q}}$, implies the result. □

Now, we derive a preliminary bound for the error associated with the temperature.

Lemma 4.3.5. *Assuming that $\theta|_{\Omega_{\mathrm{m}}} \in \mathrm{W}^{1,4}(\Omega_{\mathrm{m}})$ and that $\theta_{\mathrm{D}} \in \mathrm{P}_1(\Gamma_h)$, the following estimate holds*

$$\|\theta-\theta_h\|_{1,\Omega} \leq \alpha_{\mathrm{T}}^{-1}\big(\|R^{\mathrm{T}}\| + \bar{C}_{O_{\mathrm{T}}}\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}}(\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{1,4,\Omega_{\mathrm{m}}})\big), \tag{4.41}$$

where $R^{\mathrm{T}} : \Psi_{\infty,0} \to \mathbb{R}$ is the linear and bounded functional defined by

$$R^{\mathrm{T}}(\psi) := -A_{\mathrm{T}}(\theta_h,\psi) - O_{\mathrm{T}}^h(\mathbf{u}_h;\theta_h,\psi), \tag{4.42}$$

for all $\psi \in \Psi_{\infty,0}$, with A_{T} and O_{T}^h are given in (4.7) and (4.20), respectively.

Proof. Given $\theta_1 = \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega)$ and $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$, we let $\theta_0 = \theta - \theta_1 \in \mathrm{H}_0^1(\Omega)$ and $\theta_{h,0} = \theta_h - \theta_{h,1} \in \Psi_{h,0}$ and observe, owing to assumption $\theta|_{\Omega_{\mathrm{m}}} \in \mathrm{W}^{1,4}(\Omega_{\mathrm{m}})$ and the Sobolev embedding $\mathrm{W}^{1,4}(\Omega_{\mathrm{m}}) \hookrightarrow \mathrm{L}^\infty(\Omega_{\mathrm{m}})$, that $\theta_0 \in \Psi_{\infty,0}$. In addition, since $\theta_{\mathrm{D}}|_\Gamma \in \mathrm{P}_1(\Gamma_h)$, it is clear that $(\theta_1-\theta_{h,1})|_\Gamma = 0$, thus $(\theta-\theta_h) = (\theta_0-\theta_{h,0}) + (\theta_1-\theta_{h,1}) \in \Psi_{\infty,0}$.

Then, recalling that $A_{\mathrm{T}}(\cdot,\cdot) + O_{\mathrm{T}}(\mathbf{v};\cdot,\cdot)$ is $\Psi_{\infty,0}$-elliptic for any $\mathbf{v} \in \mathbf{V}$ (cf. (3.17)), using the fact that $O_{\mathrm{T}}(\mathbf{v};\theta,\psi) = O_{\mathrm{T}}^h(\mathbf{v};\theta,\psi)$ for all $\mathbf{v} \in \mathbf{H}$, $\theta \in \mathrm{H}^1(\Omega)$, $\psi \in \Psi_{\infty,0}$, and recalling that $\mathbf{u} \in \mathbf{V}$, we deduce that

$$\alpha_{\mathrm{T}}\|\theta-\theta_h\|_{1,\Omega}^2 \leq A_{\mathrm{T}}(\theta-\theta_h,\theta-\theta_h) + O_{\mathrm{T}}^h(\mathbf{u};\theta-\theta_h,\theta-\theta_h),$$

which after simple algebraic manipulation, implies

$$\alpha_{\mathrm{T}}\|\theta-\theta_h\|_{1,\Omega}^2 \leq R^{\mathrm{T}}(\theta-\theta_h) - O_{\mathrm{T}}^h(\mathbf{u}-\mathbf{u}_h;\theta,\theta-\theta_h) + O_{\mathrm{T}}^h(\mathbf{u}-\mathbf{u}_h;\theta-\theta_h,\theta-\theta_h), \tag{4.43}$$

where $R^{\mathrm{T}} \in \Psi_{\infty,0}'$ is given by (4.42). In this way, recalling from Lemmas 3.2.2 and 3.3.2 that $O_{\mathrm{T}}^h(\mathbf{u};\theta-\theta_h,\theta-\theta_h) = 0$ and noticing that a slight adaptation of Lemma 3.4.1 implies

$$|O_{\mathrm{T}}^h(\mathbf{u}-\mathbf{u}_h;\theta,\theta-\theta_h)| \leq \bar{C}_{O_{\mathrm{T}}}\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}}(\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{1,4,\Omega_{\mathrm{m}}})\|\theta-\theta_h\|_{1,\Omega},$$

from (4.43) we obtain the result. □

Using the previous lemmas now we derive a preliminary upper bound for the total error.

Lemma 4.3.6. *Assume the same hypotheses of Lemmas 4.3.4 and 4.3.5. Assume further that the following estimate holds*

$$C_1\gamma_{\mathbf{g}}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + C_2\gamma_{\mathbf{g}}\|\theta\|_{1,4,\Omega_{\mathrm{m}}} < 1, \tag{4.44}$$

where $\gamma_{\mathbf{g}} := \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$ *and* C_1, $C_2 > 0$ *are defined in* (4.47). *Then, there exists* $C > 0$, *independent of the mesh-size, such that*

$$\|(\mathbf{u},(p,\lambda),\theta) - (\mathbf{u}_h,(p_h,\lambda_h),\theta_h)\| \le C(\|R^{\mathrm{F}}\| + \|R^{\mathrm{T}}\|), \tag{4.45}$$

where R^{F} *and* R^{T} *are the linear and bounded functional defined in* (4.39) *and* (4.42)*, respectively.*

Proof. We begin the proof by replacing (4.41) back into (4.38), to obtain

$$\begin{aligned}
&\|(\mathbf{u}-\mathbf{u}_h,(p-p_h,\lambda-\lambda_h))\|_{\mathbf{H}\times\mathrm{Q}} \\
&\qquad \le 2\gamma^{-1}\big[\|R^{\mathrm{F}}\| + C_D\gamma_{\mathbf{g}}\alpha_{\mathrm{T}}^{-1}(\|R^{\mathrm{T}}\| + \bar{C}_{O_{\mathrm{T}}}\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}}(\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{1,4,\Omega_{\mathrm{m}}}))\big].
\end{aligned}$$

In turn, we note that owing to the a priori estimates for θ in (4.14) and estimate (4.10b), there holds

$$\|\theta\|_{1,\Omega_{\mathrm{f}}} \le \|\theta\|_{1,\Omega} \le (C_\theta+1)C_{\mathrm{lift},2}\delta^{-3/4}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma},$$

which when combined with the previous estimate yields

$$\begin{aligned}
\|(\mathbf{u},(p,\lambda)) - (\mathbf{u}_h,(p_h,\lambda_h))\|_{\mathbf{H}\times\mathrm{Q}} &\le 2\gamma^{-1}\|R^{\mathrm{F}}\| + 2\gamma^{-1}\gamma_{\mathbf{g}}C_D\alpha_{\mathrm{T}}^{-1}\|R^{\mathrm{T}}\| \\
&\quad + \big[C_1\gamma_{\mathbf{g}}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + C_2\gamma_{\mathbf{g}}\|\theta\|_{1,4,\Omega_{\mathrm{m}}}\big]\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}},
\end{aligned} \tag{4.46}$$

where

$$\begin{aligned}
C_1 &:= 2\gamma^{-1}C_D\alpha_{\mathrm{T}}^{-1}\bar{C}_{O_{\mathrm{T}}}(C_\theta+1)C_{\mathrm{lift},2}\delta^{-3/4}(1+\delta^2)^{1/2}, \\
C_2 &:= 2\gamma^{-1}C_D\alpha_{\mathrm{T}}^{-1}\bar{C}_{O_{\mathrm{T}}}.
\end{aligned} \tag{4.47}$$

In this way, from (4.46) and (4.44), we readily obtain

$$\|(\mathbf{u},(p,\lambda)) - (\mathbf{u}_h,(p_h,\lambda_h))\|_{\mathbf{H}\times\mathrm{Q}} \le 2\gamma^{-1}\|R^{\mathrm{F}}\| + 2\gamma^{-1}\gamma_{\mathbf{g}}C_D\alpha_{\mathrm{T}}^{-1}\|R^{\mathrm{T}}\|. \tag{4.48}$$

In addition, using he fact that $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}} \le \|(\mathbf{u},(p,\lambda)) - (\mathbf{u}_h,(p_h,\lambda_h))\|_{\mathbf{H}\times\mathrm{Q}}$, from (4.41), (4.48) and (4.44), we deduce that

$$\|\theta-\theta_h\|_{1,\Omega} \le 2C_D^{-1}\gamma_{\mathbf{g}}^{-1}\|R^{\mathrm{F}}\| + (2\alpha_{\mathrm{T}}^{-1}+1)\|R^{\mathrm{T}}\|, \tag{4.49}$$

which together with (4.48), implies the result. □

According to the upper bound (4.45), we observe in advance that to obtain the reliability of the estimator, it suffices to estimate $\|R^{\mathrm{F}}\|$ and $\|R^{\mathrm{T}}\|$. This will be the focus of the next section.

Upper bounds for $\|R^{\mathrm{F}}\|$ and $\|R^{\mathrm{T}}\|$

We begin by deriving a suitable upper bound for R^{F} (cf. (4.39)). To this end, note that from the first and second equations of (4.18), it follows that $R^{\mathrm{F}}(\mathbf{v}_h,(q_h,\xi_h)) = 0$ for all $(\mathbf{v}_h,(q_h,\xi_h)) \in \mathbf{H}_h \times \mathrm{Q}_h$. Consequently, for any given $((\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{m}}),(q,\xi)) \in \mathbf{H} \times \mathrm{Q}$ and $(\mathbf{v}_{h,\mathrm{f}},\mathbf{v}_{h,\mathrm{m}}) \in \mathbf{H}_h$, we can express R^{F} as:

$$\begin{aligned} R^{\mathrm{F}}\big((\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{m}}),(q,\xi)\big) =&\ R^{\mathrm{F}}\big(((\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{m}}),(q,\xi)) - ((\mathbf{v}_{h,\mathrm{f}},\mathbf{v}_{h,\mathrm{m}}),(0,0))\big) \\ =&\ R^{\mathrm{F}}_1(\mathbf{v}_{\mathrm{f}} - \mathbf{v}_{h,\mathrm{f}}) + R^{\mathrm{F}}_2(\mathbf{v}_{\mathrm{m}} - \mathbf{v}_{h,\mathrm{m}}) + R^{\mathrm{F}}_3(q,\xi), \end{aligned} \tag{4.50}$$

where

$$\begin{aligned} R^{\mathrm{F}}_1(\mathbf{v}_{\mathrm{f}}) :=&- A_{\mathrm{F}}(\mathbf{u}_h,(\mathbf{v}_{\mathrm{f}},\mathbf{0})) - O^h_{\mathrm{F}}(\mathbf{u}_{h,\mathrm{f}};\mathbf{u}_{h,\mathrm{f}},\mathbf{v}_{\mathrm{f}}) - B((\mathbf{v}_{\mathrm{f}},\mathbf{0}),(p_h,\lambda_h)) \\ &+ D(\theta_h,(\mathbf{v}_{\mathrm{f}},\mathbf{0})), \end{aligned} \tag{4.51}$$

$$R^{\mathrm{F}}_2(\mathbf{v}_{\mathrm{m}}) := - A_{\mathrm{F}}(\mathbf{u}_h,(\mathbf{0},\mathbf{v}_{\mathrm{m}})) - B((\mathbf{0},\mathbf{v}_{\mathrm{m}}),(p_h,\lambda_h)) + D(\theta_h,(\mathbf{0},\mathbf{v}_{\mathrm{m}})), \tag{4.52}$$

$$R^{\mathrm{F}}_3(q,\xi) := - B(\mathbf{u}_h,(q,\xi)), \tag{4.53}$$

for all $((\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{m}}),(q,\xi)) \in \mathbf{H} \times \mathrm{Q}$. Thus, given $((\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{m}}),(q,\xi)) \in \mathbf{H} \times \mathrm{Q}$, to bound $R^{\mathrm{F}}\big((\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{m}}),(q,\xi)\big)$, it suffices to obtain bounds for $R^{\mathrm{F}}_1(\mathbf{v}_{\mathrm{f}} - \mathbf{v}_{h,\mathrm{f}})$, $R^{\mathrm{F}}_2(\mathbf{v}_{\mathrm{m}} - \mathbf{v}_{h,\mathrm{m}})$, and $R^{\mathrm{F}}_3(q,\xi)$, with appropriate choices of $\mathbf{v}_{\mathrm{f},h}$ and $\mathbf{v}_{\mathrm{m},h}$ for the first two functionals. We start by estimating $R^{\mathrm{F}}_3(q,\xi)$.

Lemma 4.3.7. *There exists $c > 0$, independent of the mesh-size, such that*

$$|R^{\mathrm{F}}_3(q,\xi)| \le c \left\{ \sum_{T\in\mathcal{T}^{\mathrm{f}}_h} \|\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}\|^2_{0,T} + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n} - \mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n}\|^2_{0,e} \right\}^{1/2} \|(q,\xi)\|_{\mathrm{Q}},$$

for all $(q,\xi) \in \mathrm{Q}$.

Proof. From the second equation of (4.18) we first recall that $\mathrm{div}\,\mathbf{u}_{h,\mathrm{m}} = 0$ in Ω_{m}, thus

$$R^{\mathrm{F}}_3(q,\xi) = (q,\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}})_{\Omega_{\mathrm{f}}} - \langle \mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n} - \mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n},\xi\rangle_{\Sigma} \qquad \forall\,(q,\xi) \in \mathrm{Q}$$

Then, from the Cauchy-Schwarz inequality, it is easy to see that

$$\begin{aligned} |R^{\mathrm{F}}_3(q,\xi)| \le&\ \Big\{ \|\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}\|^2_{0,\Omega_{\mathrm{f}}} + \|\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n} - \mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n}\|^2_{-1/2,\Sigma} \Big\}^{1/2} \|(q,\xi)\|_{\mathrm{Q}} \\ =&\ \left\{ \sum_{T\in\mathcal{T}^{\mathrm{f}}_h} \|\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}\|^2_{0,T} + \|\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n} - \mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n}\|^2_{-1/2,\Sigma} \right\}^{1/2} \|(q,\xi)\|_{\mathrm{Q}}. \end{aligned} \tag{4.54}$$

In turn, to bound $\|\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n}-\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n}\|^2_{-1/2,\Sigma}$ in terms of local quantities, we proceed similarly to [4, Lemma 3.6] and [59, Lemma 3.2]. In, fact, we observe that $(\mathbf{u}_{h,\mathrm{f}}-\mathbf{u}_{h,\mathrm{m}})\cdot\mathbf{n}$ is $\mathrm{L}^2(\Sigma)-$orthogonal to $\Lambda_h(\Sigma)$, recall that Σ_h and Σ_{2h} are of bounded variation and apply [22, Theorem 2], to deduce that there exists $c>0$, independent of the mesh-size, such that

$$\begin{aligned}\|\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n}-\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n}\|^2_{-1/2,\Sigma} &\le c\sum_{e\in\mathcal{E}_{2h}(\Sigma)} h_e\|\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n}-\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n}\|^2_{0,e}\\ &\le c\sum_{e\in\mathcal{E}_{h}(\Sigma)} h_e\|\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n}-\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n}\|^2_{0,e}.\end{aligned}$$

This estimate and (4.54) imply the result. □

Now we take $\mathbf{v}_{h,\mathrm{f}}=\boldsymbol{\mathcal{C}}_{h,\mathrm{f}}(\mathbf{v}_\mathrm{f})$, with $\boldsymbol{\mathcal{C}}_{h,\mathrm{f}}$ being the vector version of the Clément operator defined in Section 4.3.1 and obtain a suitable estimate for $R_1^{\mathrm{F}}(\mathbf{v}_\mathrm{f}-\mathbf{v}_{h,\mathrm{f}})$.

Lemma 4.3.8. *There exists $c>0$, independent of the mesh-size, such that*

$$|R_1^{\mathrm{F}}(\mathbf{v}_\mathrm{f}-\boldsymbol{\mathcal{C}}_{h,\mathrm{f}}(\mathbf{v}_\mathrm{f}))|\le c\left\{\sum_{T\in\mathcal{T}_h^\mathrm{f}}\eta^2_{T,\mathrm{f},1}\right\}^{1/2}\|\mathbf{v}_\mathrm{f}\|_{1,\Omega_\mathrm{f}}\qquad\forall\,\mathbf{v}_\mathrm{f}\in\mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}),\tag{4.55}$$

where

$$\begin{aligned}\eta^2_{T,\mathrm{f},1}\;&:=h_T^2(\|\mathbf{div}\,\boldsymbol{\sigma}_{h,\mathrm{f}}-(\mathbf{u}_{h,\mathrm{f}}\cdot\nabla)\mathbf{u}_{h,\mathrm{f}}-\tfrac{1}{2}\mathbf{u}_{h,\mathrm{f}}\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}+\theta_h\mathbf{g}_\mathrm{f}\|^2_{0,T})\\ &+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_\mathrm{f})}h_e\|[\![\boldsymbol{\sigma}_{h,\mathrm{f}}\mathbf{n}]\!]\|^2_{0,e}+\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)}h_e\|\boldsymbol{\sigma}_{h,\mathrm{f}}\mathbf{n}+\omega_1(\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{t})\mathbf{t}+\lambda_h\mathbf{n}\|^2_{0,e},\end{aligned}\tag{4.56}$$

and $\boldsymbol{\sigma}_{h,\mathrm{f}}$ is given by (4.34).

Proof. Let $\mathbf{v}_\mathrm{f}\in\mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$ and denote by $\widehat{\mathbf{v}}_\mathrm{f}:=\mathbf{v}_\mathrm{f}-\boldsymbol{\mathcal{C}}_{h,\mathrm{f}}(\mathbf{v}_\mathrm{f})$. Since $\mathbf{v}_\mathrm{f}\in\mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, clearly $\boldsymbol{\mathcal{C}}_{h,\mathrm{f}}(\mathbf{v}_\mathrm{f})\in\mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$. In addition, from the approximation properties provided in Lemma 4.3.1, we deduce that

$$\|\widehat{\mathbf{v}}_\mathrm{f}\|_{0,T}\le c_1h_T\|\mathbf{v}_\mathrm{f}\|_{1,\Delta_\mathrm{f}(T)}\qquad\forall\,T\in\mathcal{T}_h^\mathrm{f},\tag{4.57}$$

and

$$\|\widehat{\mathbf{v}}_\mathrm{f}\|_{0,e}\le c_2h_e^{1/2}\|\mathbf{v}_\mathrm{f}\|_{1,\Delta_\mathrm{f}(e)}\qquad\forall\,e\in\mathcal{E}_h(\Omega_\mathrm{f})\cup\mathcal{E}_h(\Sigma).\tag{4.58}$$

In turn, from (4.51) and the definitions of A_F, O_F^h, B and D, it follows that

$$\begin{aligned}R_1^{\mathrm{F}}(\widehat{\mathbf{v}}_\mathrm{f})=&-(2\mu\mathbf{e}(\mathbf{u}_{h,\mathrm{f}}),\mathbf{e}(\widehat{\mathbf{v}}_\mathrm{f}))_{\Omega_\mathrm{f}}-\omega_1\langle\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{t},\widehat{\mathbf{v}}_\mathrm{f}\cdot\mathbf{t}\rangle_\Sigma-((\mathbf{u}_{h,\mathrm{f}}\cdot\nabla)\mathbf{u}_{h,\mathrm{f}},\widehat{\mathbf{v}}_\mathrm{f})_{\Omega_\mathrm{f}}\\ &-\tfrac{1}{2}(\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}},\mathbf{u}_{h,\mathrm{f}}\cdot\widehat{\mathbf{v}}_\mathrm{f})_{\Omega_\mathrm{f}}+(p_h,\mathrm{div}\,\widehat{\mathbf{v}}_\mathrm{f})_{\Omega_\mathrm{f}}-\langle\widehat{\mathbf{v}}_\mathrm{f}\cdot\mathbf{n},\lambda_h\rangle_\Sigma+(\theta_h\mathbf{g}_\mathrm{f},\widehat{\mathbf{v}}_\mathrm{f})_{\Omega_\mathrm{f}}.\end{aligned}$$

Then, noting that $p_h \mathrm{div}\, \widehat{\mathbf{v}}_{\mathrm{f}} = p_h \mathbf{I} : \nabla \widehat{\mathbf{v}}_{\mathrm{f}}$, using the symmetry of $\mathbf{e}(\cdot)$, and recalling the definition of $\boldsymbol{\sigma}_{h,\mathrm{f}}$ (cf. (4.34)), we find that

$$\begin{aligned} R_1^{\mathrm{F}}(\widehat{\mathbf{v}}_{\mathrm{f}}) = \quad & -(\boldsymbol{\sigma}_{h,\mathrm{f}}, \nabla \widehat{\mathbf{v}}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} - \omega_1 \langle \mathbf{u}_{h,\mathrm{f}} \cdot \mathbf{t}, \widehat{\mathbf{v}}_{\mathrm{f}} \cdot \mathbf{t} \rangle_{\Sigma} - ((\mathbf{u}_{h,\mathrm{f}} \cdot \nabla)\mathbf{u}_{h,\mathrm{f}}, \widehat{\mathbf{v}}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} \\ & -\tfrac{1}{2}(\mathrm{div}\, \mathbf{u}_{h,\mathrm{f}}, \mathbf{u}_{h,\mathrm{f}} \cdot \widehat{\mathbf{v}}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} - \langle \widehat{\mathbf{v}}_{\mathrm{f}} \cdot \mathbf{n}, \lambda_h \rangle_{\Sigma} + (\theta_h \mathbf{g}_{\mathrm{f}}, \widehat{\mathbf{v}}_{\mathrm{f}})_{\Omega_{\mathrm{f}}}. \end{aligned} \tag{4.59}$$

Then, integrating by parts on each $T \in \mathcal{T}_h^{\mathrm{f}}$ and observing that $\widehat{\mathbf{v}}_{\mathrm{f}} = \mathbf{0}$ on Γ_{f}, we deduce that

$$\begin{aligned} -(\boldsymbol{\sigma}_{h,\mathrm{f}}, \nabla \widehat{\mathbf{v}}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} & - ((\mathbf{u}_{h,\mathrm{f}} \cdot \nabla)\mathbf{u}_{h,\mathrm{f}}, \widehat{\mathbf{v}}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} - \tfrac{1}{2}(\mathrm{div}\, \mathbf{u}_{h,\mathrm{f}}, \mathbf{u}_{h,\mathrm{f}} \cdot \widehat{\mathbf{v}}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} + (\theta_h \mathbf{g}_{\mathrm{f}}, \widehat{\mathbf{v}}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} \\ & = \sum_{T \in \mathcal{T}_h^{\mathrm{f}}} (\mathbf{div}\, \boldsymbol{\sigma}_{h,\mathrm{f}} - (\mathbf{u}_{h,\mathrm{f}} \cdot \nabla)\mathbf{u}_{h,\mathrm{f}} - \tfrac{1}{2}\mathbf{u}_{h,\mathrm{f}} \mathrm{div}\, \mathbf{u}_{h,\mathrm{f}} + \theta_h \mathbf{g}_{\mathrm{f}}, \widehat{\mathbf{v}}_{\mathrm{f}})_T \\ & \quad - \sum_{e \in \mathcal{E}_h(\Omega_{\mathrm{f}})} ([\![\boldsymbol{\sigma}_{h,\mathrm{f}} \mathbf{n}]\!], \widehat{\mathbf{v}}_{\mathrm{f}})_e - \sum_{e \in \mathcal{E}_h(\Sigma)} (\boldsymbol{\sigma}_{h,\mathrm{f}} \mathbf{n}, \widehat{\mathbf{v}}_{\mathrm{f}})_e, \end{aligned}$$

which, replaced back into (4.59), gives

$$\begin{aligned} R_1^{\mathrm{F}}(\widehat{\mathbf{v}}_{\mathrm{f}}) = & \sum_{T \in \mathcal{T}_h^{\mathrm{f}}} (\mathbf{div}\, \boldsymbol{\sigma}_{h,\mathrm{f}} - (\mathbf{u}_{h,\mathrm{f}} \cdot \nabla)\mathbf{u}_{h,\mathrm{f}} - \tfrac{1}{2}\mathbf{u}_{h,\mathrm{f}} \mathrm{div}\, \mathbf{u}_{h,\mathrm{f}} + \theta_h \mathbf{g}_{\mathrm{f}}, \widehat{\mathbf{v}}_{\mathrm{f}})_T \\ & - \sum_{e \in \mathcal{E}_h(\Omega_{\mathrm{f}})} ([\![\boldsymbol{\sigma}_{h,\mathrm{f}} \mathbf{n}]\!], \widehat{\mathbf{v}}_{\mathrm{f}})_e - \sum_{e \in \mathcal{E}_h(\Sigma)} \left(\boldsymbol{\sigma}_{h,\mathrm{f}} \mathbf{n} + \omega_1 (\mathbf{u}_{h,\mathrm{f}} \cdot \mathbf{t})\mathbf{t} + \lambda_h \mathbf{n}, \widehat{\mathbf{v}}_{\mathrm{f}}\right)_e. \end{aligned}$$

In this way, applying the Cauchy–Schwarz inequality, the latter implies

$$\begin{aligned} |R_1^{\mathrm{F}}(\widehat{\mathbf{v}}_{\mathrm{f}})| \le & \sum_{T \in \mathcal{T}_h^{\mathrm{f}}} \|\mathbf{div}\, \boldsymbol{\sigma}_{h,\mathrm{f}} - (\mathbf{u}_{h,\mathrm{f}} \cdot \nabla)\mathbf{u}_{h,\mathrm{f}} - \tfrac{1}{2}\mathbf{u}_{h,\mathrm{f}} \mathrm{div}\, \mathbf{u}_{h,\mathrm{f}} + \theta_h \mathbf{g}_{\mathrm{f}}\|_{0,T} \|\widehat{\mathbf{v}}_{\mathrm{f}}\|_{0,T} \\ & + \sum_{e \in \mathcal{E}_h(\Omega_{\mathrm{f}})} \|[\![\boldsymbol{\sigma}_{h,\mathrm{f}} \mathbf{n}]\!]\|_{0,e} \|\widehat{\mathbf{v}}_{\mathrm{f}}\|_{0,e} \\ & + \sum_{e \in \mathcal{E}_h(\Sigma)} \|\boldsymbol{\sigma}_{h,\mathrm{f}} \mathbf{n} + \omega_1 (\mathbf{u}_{h,\mathrm{f}} \cdot \mathbf{t})\mathbf{t} + \lambda_h \mathbf{n}\|_{0,e} \|\widehat{\mathbf{v}}_{\mathrm{f}}\|_{0,e}, \end{aligned}$$

which together with estimates (4.57) and (4.58), the fact that the numbers of triangles in $\Delta_{\mathrm{f}}(T)$ and $\Delta_{\mathrm{f}}(e)$ are bounded implies (4.55). □

Now, given $\mathbf{v}_{\mathrm{m}} \in \mathbf{H}(\mathrm{div}; \Omega_{\mathrm{m}})$, our next goal is to derive the upper bound for $R_2^{\mathrm{F}}(\mathbf{v}_{\mathrm{m}} - \mathbf{v}_{h,\mathrm{m}})$, for a suitable choice of $\mathbf{v}_{h,\mathrm{m}}$. To do this, we assume that the hypotheses of Lemma 4.3.3 hold and let $\mathbf{z} \in \mathbf{H}^1(\Omega_{\mathrm{m}})$ and $\zeta \in \mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})$, such that $\mathbf{v}_{\mathrm{m}} = \mathbf{z} + \mathrm{curl}\,(\zeta)$. Then, we choose $\mathbf{v}_{h,\mathrm{m}} = \Xi_{h,\mathrm{m}}(\mathbf{v}_{\mathrm{m}})$, with

$$\Xi_{h,\mathrm{m}}(\mathbf{v}_{\mathrm{m}}) := \Pi^{RT}(\mathbf{z}) + \mathrm{curl}\,(\mathcal{C}_{h,\mathrm{m}}(\zeta)) \in \mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}}), \tag{4.60}$$

where Π^{RT} and $\mathcal{C}_{h,\mathrm{m}}$ are the Raviart–Thomas and Clément interpolators, respectively and observe that

$$R_2^{\mathrm{F}}(\mathbf{v}_{\mathrm{m}} - \Xi_{h,\mathrm{m}}(\mathbf{v}_{\mathrm{m}})) = R_2^{\mathrm{F}}(\mathbf{z} - \Pi^{RT}(\mathbf{z})) + R_2^{\mathrm{F}}(\mathrm{curl}\,(\zeta - \mathcal{C}_{h,\mathrm{m}}(\zeta))). \tag{4.61}$$

Then, to bound $R_2^{\mathrm{F}}(\mathbf{v}_{\mathrm{m}} - \Xi_{h,\mathrm{m}}(\mathbf{v}_{\mathrm{m}}))$, in what follows we proceed similarly to [59, Lemma 3.6] and derive estimates for $R_2^{\mathrm{F}}(\mathbf{z} - \Pi^{RT}(\mathbf{z}))$ and $R_2^{\mathrm{F}}(\mathrm{curl}\,(\zeta - \mathcal{C}_{h,\mathrm{m}}(\zeta)))$, separately. This is addressed in the following two lemmas.

Lemma 4.3.9. *There exists $c > 0$, independent of the mesh-size, such that*

$$|R_2^{\mathrm{F}}(\mathbf{z} - \Pi^{RT}(\mathbf{z}))| \leq c \left\{ \sum_{T \in \mathcal{T}_h^{\mathrm{m}}} \eta_{T,\mathrm{m},1}^2 \right\}^{1/2} \|\mathbf{z}\|_{1,\Omega_{\mathrm{m}}} \qquad \forall\, \mathbf{z} \in \mathbf{H}^1(\Omega_{\mathrm{m}}), \tag{4.62}$$

where

$$\eta_{T,\mathrm{m},1}^2 := \quad h_T^2 \| - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}} + \theta_h \mathbf{g}_{\mathrm{m}}\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \|\lambda_h - p_h\|_{0,e}^2. \tag{4.63}$$

Proof. Let $\mathbf{z} \in \mathbf{H}^1(\Omega_{\mathrm{m}})$ and define $\widehat{\mathbf{z}} := \mathbf{z} - \Pi^{RT}(\mathbf{z})$. Then, by the approximation properties of the Raviart–Thomas operator given in Lemma 4.3.2, we get

$$\|\widehat{\mathbf{z}}\|_{0,T} \leq c h_T \|\mathbf{z}\|_{1,T} \qquad \forall\, T \in \mathcal{T}_h^{\mathrm{m}}, \tag{4.64}$$

and

$$\|\widehat{\mathbf{z}} \cdot \mathbf{n}\|_{0,e} \leq c h_e^{1/2} \|\mathbf{z}\|_{1,T_e} \qquad \forall\, \text{edge } e \text{ of } \mathcal{T}_h^{\mathrm{m}}. \tag{4.65}$$

Now, according to the definitions of A_{F}, B and D, it follows from (4.52) that

$$R_2^{\mathrm{F}}(\widehat{\mathbf{z}}) = -(\mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}, \widehat{\mathbf{z}})_{\Omega_{\mathrm{m}}} + (p_h, \mathrm{div}\,\widehat{\mathbf{z}})_{\Omega_{\mathrm{m}}} + \langle \widehat{\mathbf{z}} \cdot \mathbf{n}, \lambda_h \rangle_{\Sigma} + (\theta_h \mathbf{g}_{\mathrm{m}}, \widehat{\mathbf{z}})_{\Omega_{\mathrm{m}}}.$$

Then, integrating by parts the term $(p_h, \mathrm{div}\,\widehat{\mathbf{z}})_T$ on each $T \in \mathcal{T}_h^{\mathrm{m}}$, using the commutative diagram (4.28), the fact that $(p_h|_T)|_e \in \mathrm{P}_0(e)$ for each $e \in \mathcal{E}(T)$, and for each $T \in \mathcal{T}_h^{\mathrm{m}}$, and the characterization of Π^{RT} given in (4.27), we obtain

$$R_2^{\mathrm{F}}(\widehat{\mathbf{z}}) = \sum_{T \in \mathcal{T}_h^{\mathrm{m}}} (-\mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}} + \theta_h \mathbf{g}_{\mathrm{m}}, \widehat{\mathbf{z}})_T + \sum_{e \in \mathcal{E}_h(\Sigma)} (\widehat{\mathbf{z}} \cdot \mathbf{n}, \lambda_h - p_h)_e,$$

which combined with the Cauchy–Schwarz inequality, the fact that the numbers of triangles are bounded, and estimates (4.64) and (4.65), implies the result. $\square$

Lemma 4.3.10. *There exists $c > 0$, independent of the mesh-size, such that*

$$|R_2^{\mathrm{F}}(\mathrm{curl}\,(\zeta - \mathcal{C}_{h,\mathrm{m}}(\zeta))| \leq c \left\{ \sum_{T \in \mathcal{T}_h^{\mathrm{m}}} \eta_{T,\mathrm{m},2}^2 \right\}^{1/2} \|\zeta\|_{1,\Omega_{\mathrm{m}}} \qquad \forall\, \zeta \in \mathrm{H}_{\Gamma_{\mathrm{m}}}^1(\Omega_{\mathrm{m}}), \tag{4.66}$$

where

$$\eta_{T,\mathrm{m},2}^2 := h_T^2\|\mathrm{rot}\,(\theta_h \mathbf{g}_\mathrm{m} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}})\|_{0,T} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_\mathrm{m})} h_e\|[\![(\theta_h\mathbf{g}_\mathrm{m} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}})\cdot\mathbf{t}]\!]\|_{0,e} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\|\theta_h\mathbf{g}_\mathrm{m}\cdot\mathbf{t} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{t} - \tfrac{d\lambda_h}{d\mathbf{t}}\|_{0,e}. \tag{4.67}$$

Proof. Let $\zeta \in \mathrm{H}^1_{\Gamma_\mathrm{m}}(\Omega_\mathrm{m})$ and define $\widehat{\beta} := \zeta - \mathcal{C}_{h,\mathrm{m}}(\zeta) \in \mathrm{H}^1_{\Gamma_\mathrm{m}}(\Omega_\mathrm{m})$, which according to the approximation properties given in Lemma 4.3.1, satisfies

$$\|\widehat{\beta}\|_{0,T} \le c_1 h_T\|\zeta\|_{1,\Delta_\mathrm{m}(T)} \qquad \forall\, T\in\mathcal{T}_h^\mathrm{m}, \tag{4.68}$$

and

$$\|\widehat{\beta}\|_{0,e} \le c_2 h_e^{1/2}\|\zeta\|_{1,\Delta_\mathrm{m}(e)} \qquad \forall\, e\in\mathcal{E}_h(\Omega_\mathrm{m})\cup\mathcal{E}_h(\Sigma). \tag{4.69}$$

In turn, from the definitions of A_F, B and D, and the fact that $\mathrm{div}\,(\mathrm{curl}\,(\widehat{\beta})) = 0$ in Ω_m, from (4.52) we find that

$$R_2^\mathrm{F}(\mathrm{curl}\,(\widehat{\beta})) = -(\mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}, \mathrm{curl}\,(\widehat{\beta}))_{\Omega_\mathrm{m}} + \langle \mathrm{curl}\,(\widehat{\beta})\cdot\mathbf{n}, \lambda_h\rangle_\Sigma + (\theta_h\mathbf{g}_\mathrm{m}, \mathrm{curl}\,(\widehat{\beta}))_{\Omega_\mathrm{m}}. \tag{4.70}$$

Then, we observe that $\mathrm{curl}\,(\widehat{\beta})\cdot\mathbf{n} = -\frac{d\widehat{\beta}}{d\mathbf{t}}$, and integrate by parts the term $(\mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}, \mathrm{curl}\,(\widehat{\beta}))_T$ on each $T\in\mathcal{T}_h^\mathrm{m}$ and $\langle \mathrm{curl}\,(\widehat{\beta})\cdot\mathbf{n}, \lambda_h\rangle_\Sigma$ on Σ, to deduce that

$$\begin{aligned} R_2^\mathrm{F}(\mathrm{curl}\,(\widehat{\beta})) \;:=\; & \sum_{T\in\mathcal{T}_h^\mathrm{m}} (\mathrm{rot}\,(-\mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}} + \theta_h\mathbf{g}_\mathrm{m}), \widehat{\beta})_T \\ & + \sum_{e\in\mathcal{E}_h(\Omega_\mathrm{m})} ([\![-\mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{t} + \theta_h\mathbf{g}_\mathrm{m}\cdot\mathbf{t}]\!], \widehat{\beta})_e \\ & + \sum_{e\in\mathcal{E}_h(\Sigma)} (-\mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{t} + \theta_h\mathbf{g}_\mathrm{m}\cdot\mathbf{t} - \tfrac{d\lambda_h}{d\mathbf{t}}, \widehat{\beta})_e. \end{aligned} \tag{4.71}$$

In this way, applying the Cauchy–Schwarz inequality, estimates (4.68) and (4.69) and the fact that the numbers of triangles in $\Delta_\mathrm{m}(T)$ and $\Delta_\mathrm{m}(e)$ are bounded, we conclude from (4.71) the required estimate (4.66). □

As a direct consequence of lemmas 4.3.9 and 4.3.10, and the stability of the Helmholtz decomposition (4.29), we obtain the following result.

Lemma 4.3.11. *Assume that the hypotheses of Lemma 4.3.3 hold. Then, there exists $c > 0$, independent of the mesh-size, such that*

$$|R_2^\mathrm{F}(\mathbf{v}_\mathrm{m} - \Xi_{h,\mathrm{m}}(\mathbf{v}_\mathrm{m}))| \le c\left\{\sum_{T\in\mathcal{T}_h^\mathrm{m}} \eta_{T,\mathrm{m},1}^2 + \eta_{T,\mathrm{m},2}^2\right\}^{1/2} \|\mathbf{v}_\mathrm{m}\|_{\mathrm{div};\Omega_\mathrm{m}}, \tag{4.72}$$

for all $\mathbf{v}_{\rm m} \in \mathbf{H}_{\Gamma_{\rm m}}(\mathrm{div};\Omega_{\rm m})$*, where* $\Xi_{h,\rm m}(\mathbf{v}_{\rm m})$ *is defined as in* (4.60) *and* $\eta^2_{T,\rm m,1}$ *and* $\eta^2_{T,\rm m,2}$ *are given by* (4.63) *and* (4.67)*, respectively.*

Now we are in position of establishing the desired estimate for $\|R^{\rm F}\|$.

Lemma 4.3.12. *Assume that the hypotheses of Lemma 4.3.3 hold. Then there exists* $C > 0$*, such that*

$$
\begin{aligned}
\|R^{\rm F}\| = & \sup_{(\mathbf{v},(q,\xi))\in\mathbf{H}\times\mathrm{Q}\backslash\{\mathbf{0}\}} \frac{|R^{\rm F}((\mathbf{v},(q,\xi)))|}{\|(\mathbf{v},(q,\xi))\|_{\mathbf{H}\times\mathrm{Q}}} \\
\leq\ & C\Big\{ \sum_{T\in\mathcal{T}_h^{\rm f}} \big\{\|\mathrm{div}\,\mathbf{u}_{h,\rm f}\|^2_{0,T} + \eta^2_{T,\rm f,1}\big\} \\
& + \sum_{T\in\mathcal{T}_h^{\rm m}} \big\{\|\mathrm{div}\,\mathbf{u}_{h,\rm m}\|^2_{0,T} + \eta^2_{T,\rm m,1} + \eta^2_{T,\rm m,2}\big\} \\
& + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\mathbf{u}_{h,\rm f}\cdot\mathbf{n} - \mathbf{u}_{h,\rm m}\cdot\mathbf{n}\|^2_{0,e}\Big\}^{1/2},
\end{aligned}
\tag{4.73}
$$

with $\eta_{T,\rm f,1}$*,* $\eta_{T,\rm m,1}$ *and* $\eta_{T,\rm m,2}$ *defined in* (4.56) (4.63) *and* (4.67)*, respectively.*

Proof. Given, $((\mathbf{v}_{\rm f},\mathbf{v}_{\rm m}),(q,\xi)) \in \mathbf{H}\times\mathrm{Q}\backslash\{\mathbf{0}\}$, let us recall that $R^{\rm F}$ satisfies (see (4.50))

$$
R^{\rm F}\big((\mathbf{v}_{\rm f},\mathbf{v}_{\rm m}),(q,\xi)\big) = R^{\rm F}_1(\mathbf{v}_{\rm f} - \boldsymbol{\mathcal{C}}_{h,\rm f}(\mathbf{v}_{\rm f})) + R^{\rm F}_2(\mathbf{v}_{\rm m} - \Xi_{h,\rm m}(\mathbf{v}_{\rm m})) + R^{\rm F}_3(q,\xi).
$$

Then, the proof follows from the latter and lemmas 4.3.7, 4.3.8 and 4.3.11. We omit further details. □

Now, we turn to derive and upper bound for $\|R^{\rm T}\|$. This result is established in the following lemma.

Lemma 4.3.13. *There exists* $c > 0$*, independent of the mesh-size, such that*

$$
\begin{aligned}
\|R^{\rm T}\| &= \sup_{\psi\in\Psi_{\infty,0}} \frac{|R^{\rm T}(\psi)|}{\|\psi\|_{1,\Omega}} \\
&\leq c\left\{ \sum_{T\in\mathcal{T}_h^{\rm f}} \eta^2_{T,\rm f,2} + \sum_{T\in\mathcal{T}_h^{\rm m}} \eta^2_{T,\rm m,3} + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\kappa_{\rm f}\nabla\theta_{h,\rm f}\cdot\mathbf{n} - \kappa_{\rm m}\nabla\theta_{h,\rm m}\cdot\mathbf{n}\|^2_{0,e} \right\}^{1/2},
\end{aligned}
\tag{4.74}
$$

where

$$
\eta^2_{T,\rm f,2} := h_T^2\|\kappa_{\rm f}\Delta\theta_h - \mathbf{u}_{h,\rm f}\cdot\nabla\theta_h - \tfrac{1}{2}\theta_h\,\mathrm{div}\,\mathbf{u}_{h,\rm f}\|^2_{0,T} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\rm f})} h_e\|[\![\kappa_{\rm f}\nabla\theta_h\cdot\mathbf{n}]\!]\|^2_{0,e},
\tag{4.75}
$$

and

$$\eta^2_{T,\mathrm{m},3} := h_T^2 \|\kappa_\mathrm{m}\Delta\theta_h - \mathbf{u}_{h,\mathrm{m}}\cdot\nabla\theta_h\|^2_{0,T} + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_\mathrm{m})} h_e \|[\![\kappa_\mathrm{m}\nabla\theta_h\cdot\mathbf{n}]\!]\|^2_{0,e}. \tag{4.76}$$

Proof. We begin the proof by noticing that $R^\mathrm{T}(\psi_h) = 0$ for all $\psi_h \in \Psi_h$. Then, we let $\psi \in \Psi_{\infty,0}$, and observe that for all $\psi_h \in \Psi_h$, there holds

$$R^\mathrm{T}(\psi) = R^\mathrm{T}(\psi - \psi_h) = -A_\mathrm{T}(\theta_h, \psi-\psi_h) - O^h_\mathrm{T}(\mathbf{u}_h;\theta_h,\psi-\psi_h). \tag{4.77}$$

In turn, denoting by $\widehat{\psi} := \psi - \mathcal{C}_h(\psi)$, where $\mathcal{C}_h$ is the Clément operator, from the approximation properties of $\mathcal{C}_h$ in Lemma 4.3.1, it is clear that $\widehat{\psi}$ satisfies

$$\|\widehat{\psi}\|_{0,T} \le c_1 h_T \|\widehat{\psi}\|_{1,\Delta(T)} \qquad \forall\, T\in\mathcal{T}_h, \tag{4.78}$$

and

$$\|\widehat{\psi}\|_{0,e} \le c_2 h_e^{1/2} \|\widehat{\psi}\|_{1,\Delta(e)} \qquad \forall\, e\in\mathcal{E}_h(\Omega). \tag{4.79}$$

Now, according to the definitions of A_T and O^h_T, from (4.42) and (4.77), it follows that

$$\begin{aligned} R^\mathrm{T}(\psi) = R^\mathrm{T}(\widehat{\psi}) = &-(\kappa_\mathrm{f}\nabla\theta_h, \nabla\widehat{\psi})_{\Omega_\mathrm{f}} - (\mathbf{u}_{h,\mathrm{f}}\cdot\nabla\theta_h, \widehat{\psi})_{\Omega_\mathrm{f}} - \tfrac{1}{2}(\operatorname{div}\mathbf{u}_{h,\mathrm{f}}, \theta_h\widehat{\psi})_{\Omega_\mathrm{f}} \\ &-(\kappa_\mathrm{m}\nabla\theta_h, \nabla\widehat{\psi})_{\Omega_\mathrm{m}} - (\mathbf{u}_{h,\mathrm{m}}\cdot\nabla\theta_h, \widehat{\psi})_{\Omega_\mathrm{m}}. \end{aligned} \tag{4.80}$$

Then, integrating by parts on each $T\in\mathcal{T}_h^\star$ with $\star\in\{\mathrm{f},\mathrm{m}\}$ and observing that $\widehat{\psi}=0$ on $\Gamma = \Gamma_\mathrm{f}\cup\Gamma_\mathrm{m}$, we deduce that

$$\begin{aligned} -(\kappa_\mathrm{f}\nabla\theta_h, \nabla\widehat{\psi})_{\Omega_\mathrm{f}} - (\kappa_\mathrm{m}\nabla\theta_h, \nabla\widehat{\psi})_{\Omega_\mathrm{m}} = &\sum_{T\in\mathcal{T}_h^\mathrm{f}} (\kappa_\mathrm{f}\Delta\theta_h, \widehat{\psi})_T - \sum_{e\in\mathcal{E}_h(\Omega_\mathrm{f})} ([\![\kappa_\mathrm{f}\nabla\theta_h\cdot\mathbf{n}]\!], \widehat{\psi})_e \\ &+ \sum_{T\in\mathcal{T}_h^\mathrm{m}} (\kappa_\mathrm{m}\Delta\theta_h, \widehat{\psi})_T - \sum_{e\in\mathcal{E}_h(\Omega_\mathrm{m})} ([\![\kappa_\mathrm{m}\nabla\theta_h\cdot\mathbf{n}]\!], \widehat{\psi})_e \\ &- \sum_{e\in\mathcal{E}_h(\Sigma)} (\kappa_\mathrm{f}\nabla\theta_{h,\mathrm{f}}\cdot\mathbf{n} - \kappa_\mathrm{m}\nabla\theta_{h,\mathrm{m}}\cdot\mathbf{n}, \widehat{\psi})_e, \end{aligned}$$

which replaced back into (4.80), gives

$$\begin{aligned} R^\mathrm{T}(\psi) = &\sum_{T\in\mathcal{T}_h^\mathrm{f}} (\kappa_\mathrm{f}\Delta\theta_h - \mathbf{u}_{h,\mathrm{f}}\cdot\nabla\theta_h - \tfrac{1}{2}\theta_h\operatorname{div}\mathbf{u}_{h,\mathrm{f}}, \widehat{\psi})_T - \sum_{e\in\mathcal{E}_h(\Omega_\mathrm{f})} ([\![\kappa_\mathrm{f}\nabla\theta_h\cdot\mathbf{n}]\!], \widehat{\psi})_e \\ &+ \sum_{T\in\mathcal{T}_h^\mathrm{m}} (\kappa_\mathrm{m}\Delta\theta_h - \mathbf{u}_{h,\mathrm{m}}\cdot\nabla\theta_h, \widehat{\psi})_T - \sum_{e\in\mathcal{E}_h(\Omega_\mathrm{m})} ([\![\kappa_\mathrm{m}\nabla\theta_h\cdot\mathbf{n}]\!], \widehat{\psi})_e \\ &- \sum_{e\in\mathcal{E}_h(\Sigma)} (\kappa_\mathrm{f}\nabla\theta_{h,\mathrm{f}}\cdot\mathbf{n} - \kappa_\mathrm{m}\nabla\theta_{h,\mathrm{m}}\cdot\mathbf{n}, \widehat{\psi})_e. \end{aligned}$$

In this way, applying the Cauchy–Schwarz inequality, estimates (4.78) and (4.79), and the fact that the numbers of triangles in $\Delta_\star(T)$ and $\Delta_\star(e)$ are bounded for $\star \in \{\mathrm{f}, \mathrm{m}\}$, we deduce that

$$
\begin{aligned}
&|R^{\mathrm{T}}(\psi)| \\
&\leq c \left\{ \sum_{T\in\mathcal{T}_h^{\mathrm{f}}} \eta_{T,\mathrm{f},2}^2 + \sum_{T\in\mathcal{T}_h^{\mathrm{m}}} \eta_{T,\mathrm{m},3}^2 + \sum_{e\in\mathcal{E}_h(\Sigma)} h_e \|\kappa_{\mathrm{f}} \nabla\theta_{h,\mathrm{f}} \cdot \mathbf{n} - \kappa_{\mathrm{m}} \nabla\theta_{h,\mathrm{m}} \cdot \mathbf{n}\|_{0,e}^2 \right\}^{1/2} \|\psi\|_{1,\Omega},
\end{aligned}
$$

for all $\psi \in \Psi_{\infty,0}$, which implies the result. □

Main result

The following theorem summarizes the results obtained in the previous sections and establishes the desired reliability of the estimator (4.30).

Theorem 4.3.14. *Let* $(\mathbf{u}, (p, \lambda), \theta)$ *be the unique solution of* (4.6) *and let* $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h)$ *be a solution of* (4.18). *Assume that the same hypotheses of Lemmas 4.3.3, 4.3.4, 4.3.5 and 4.3.6 hold. Then there exists* $C_{rel} > 0$, *independent of the mesh-size, such that*

$$
\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}} + \|(p - p_h, \lambda - \lambda_h)\|_{\mathrm{Q}} + \|\theta - \theta_h\|_{1,\Omega} \leq C_{rel} \boldsymbol{\eta}. \tag{4.81}
$$

Proof. The reliability of the estimator $\boldsymbol{\eta}$ follows from lemmas 4.3.6, 4.3.12 and 4.3.13. We omit further details. □

4.3.4 Efficiency

Now we turn to prove the lower bound of (4.35). More precisely, in what follows we proof the following result.

Theorem 4.3.15. *Let* $(\mathbf{u}, (p, \lambda), \theta) = ((\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}}), (p, \lambda), \theta_0 + \theta_1) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{H}^1(\Omega)$ *be the unique solution of* (4.6) *and let* $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) = ((\mathbf{u}_{h,\mathrm{f}}, \mathbf{u}_{h,\mathrm{m}}), (p_h, \lambda_h), \theta_{h,0} + \theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ *be a solution of* (4.18). *Assume that* $\mathbf{g}_{\mathrm{f}}$ *and* $\mathbf{g}_{\mathrm{m}}$ *are sufficiently smooth. Then, there exists* $C_{eff} > 0$, *independent of the mesh-size, such that*

$$
C_{eff} \boldsymbol{\eta} \leq \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}} + \|(p - p_h, \lambda - \lambda_h)\|_{\mathrm{Q}} + \|\theta - \theta_h\|_{1,\Omega} + \mathrm{h.o.t.}, \tag{4.82}
$$

where h.o.t. *denotes the higher order terms.*

Remark 4.3.1. *We observe here that only global efficiency is achieved. However, as we will show in Lemma 4.3.27, local efficiency can be obtained by assuming additional regularity on* λ.

In order to prove (4.82), we proceed similarly to [4] and [43] (see also [59] and [24]), using results from [22, 23], and applying inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions. We begin by providing the following preliminary result, which consists of the converse derivation of (4.6).

Theorem 4.3.16. *Let* $(\mathbf{u},(p,\lambda),\theta) = ((\mathbf{u}_\mathrm{f},\mathbf{u}_\mathrm{m}),(p,\lambda),\theta) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{H}^1(\Omega)$ *be the unique solution of* (4.6). *Then* $\operatorname{div}\mathbf{u}_\mathrm{f} = 0$ *in* Ω_f, $\operatorname{div}\mathbf{u}_\mathrm{m} = 0$ *in* Ω_m *and* $\mathbf{u}_\mathrm{f}\cdot\mathbf{n} = \mathbf{u}_\mathrm{m}\cdot\mathbf{n}$ *on* Σ. *In addition, defining* $p_\mathrm{f} := p|_{\Omega_\mathrm{f}}$, $p_\mathrm{m} := p|_{\Omega_\mathrm{m}}$, $\theta_\mathrm{f} := \theta|_{\Omega_\mathrm{f}}$, $\theta_\mathrm{m} := \theta|_{\Omega_\mathrm{m}}$, *and* $\boldsymbol{\sigma}_\mathrm{f} := 2\mu\mathbf{e}(\mathbf{u}_\mathrm{f}) - p_\mathrm{f}\mathbf{I} \in \mathbb{H}(\mathbf{div};\Omega_\mathrm{f})$, *there hold* $p_\mathrm{m} \in \mathrm{H}^1(\Omega_\mathrm{m})$, $\theta_\mathrm{f} = \theta_\mathrm{m}$ *on* Σ, $\kappa_\mathrm{f}\nabla\theta_\mathrm{f}\cdot\mathbf{n} = \kappa_\mathrm{m}\nabla\theta_\mathrm{m}\cdot\mathbf{n}$ *on* Σ, $\boldsymbol{\sigma}_\mathrm{f}\,\mathbf{n} + \omega_1(\mathbf{u}_\mathrm{f}\cdot\mathbf{t})\mathbf{t} + \lambda\mathbf{n} = 0$ *on* Σ, $\lambda = p_\mathrm{m}$ *on* Σ, $-\mathbf{div}\,\boldsymbol{\sigma}_\mathrm{f} + (\mathbf{u}_\mathrm{f}\cdot\nabla)\mathbf{u}_\mathrm{f} - \mathbf{g}_\mathrm{f}\theta_\mathrm{f} = \mathbf{0}$ *in* Ω_f, $-\kappa_\mathrm{f}\Delta\theta_\mathrm{f} + \mathbf{u}_\mathrm{f}\cdot\nabla\theta_\mathrm{f} = 0$ *in* Ω_f, $\mathbf{K}^{-1}\mathbf{u}_\mathrm{m} + \nabla p_\mathrm{m} - \mathbf{g}_\mathrm{m}\theta_\mathrm{m} = \mathbf{0}$ *in* Ω_m, *and* $-\kappa_\mathrm{m}\Delta\theta_\mathrm{m} + \mathbf{u}_\mathrm{m}\cdot\nabla\theta_\mathrm{m} = 0$ *in* Ω_m.

Proof. It is clear that the identities $\operatorname{div}\mathbf{u}_\mathrm{f} = 0$ in Ω_f, $\operatorname{div}\mathbf{u}_\mathrm{m} = 0$ in Ω_m and $\mathbf{u}_\mathrm{f}\cdot\mathbf{n} = \mathbf{u}_\mathrm{m}\cdot\mathbf{n}$ on Σ follow from the second equation of (4.6). In addition, $\theta_\mathrm{f} = \theta_\mathrm{m}$ on Σ follows from the fact that $\theta \in \mathrm{H}^1(\Omega)$. The derivation of the rest of the identities follows from the first and third equations of (4.6), considering suitable test functions and integrating by parts backwardly. We omit further details. □

In what follows we bound each local term defining $\boldsymbol{\eta}$ (cf. (4.30)) in term of local errors. We begin with the estimates for the zero order terms appearing in the definition of $\eta^2_{T,\mathrm{f}}$ and $\eta^2_{T,\mathrm{m}}$.

Lemma 4.3.17. *For all* $T \in \mathcal{T}_h^\mathrm{f}$ *there holds*

$$\|\operatorname{div}\mathbf{u}_{h,\mathrm{f}}\|_{0,T} \le \sqrt{2}|\mathbf{u}_\mathrm{f} - \mathbf{u}_{h,\mathrm{f}}|_{1,T}. \tag{4.83}$$

Proof. It suffices to recall, as established in Theorem 4.3.16, that $\operatorname{div}\mathbf{u}_\mathrm{f} = 0$ in Ω_f. □

To derive the upper bounds for the remaining terms defining the global a posteriori error estimator $\boldsymbol{\eta}$, we need to introduce further notations and preliminary results.

Given $T \in \mathcal{T}_h^\mathrm{f} \cup \mathcal{T}_h^\mathrm{m}$ and $e \in \mathcal{E}(T)$, we let φ_T and φ_e be the usual triangle-bubble and edge-bubble functions, respectively. In particular, φ_T satisfies $\varphi_T \in \mathrm{P}_3(T)$, $\operatorname{supp}(\varphi_T) \subseteq T$, $\varphi_T = 0$ on ∂T, and $0 \le \varphi_T \le 1$ in T. Similarly, $\varphi_e|_T \in \mathrm{P}_2(T)$, $\operatorname{supp}(\varphi_e) \subseteq \omega_e := \cup\{T' \in \mathcal{T}_h : e \in \mathcal{E}(T')\}$, $\varphi_e = 0$ on $\partial T \setminus e$, and $0 \le \varphi_e \le 1$ in ω_e. We also recall from [100] that, given $k \in \mathbb{N}\cup\{0\}$, there exists an extension operator $L : \mathrm{C}(e) \to \mathrm{C}(T)$ that satisfies $L(p) \in \mathrm{P}_k(T)$ and $L(p)|_e = p$ for all $p \in \mathrm{P}_k(e)$. A corresponding vector version of L, that is the componentwise application of L, is denoted by $\mathbf{L}$.

Additional properties of φ_T, φ_e, and L are collected in the following lemma (see [100, Lemma 1.3]).

Lemma 4.3.18. *Given $k \in \mathbb{N} \cup \{0\}$, there exist positive constants c_1, c_2, and c_3, depending only on k and the shape-regularity of the triangulations (minimum angle condition), such that for each triangle T and $e \in \mathcal{E}(T)$, there hold*

$$\|q\|_{0,T}^2 \leq c_1 \|\varphi_T^{1/2} q\|_{0,T}^2 \qquad \forall\, q \in \mathrm{P}_k(T), \tag{4.84}$$

$$\|p\|_{0,e}^2 \leq c_2 \|\varphi_e^{1/2} p\|_{0,e}^2 \qquad \forall\, p \in \mathrm{P}_k(e), \tag{4.85}$$

$$\|\varphi_e^{1/2} L(p)\|_{0,T}^2 \leq c_3 h_e \|p\|_{0,e}^2 \qquad \forall\, p \in \mathrm{P}_k(e). \tag{4.86}$$

In what follows we will also make use of the following inverse inequalities (see [47, Lemma 1.138])

$$|v|_{l,p,T} \leq c h_T^{-(l+1)+2/p} \|v\|_{0,T} \quad \forall\, v \in \mathrm{P}_k(T), \text{ with } p \geq 1 \text{ and } l \geq 0, \tag{4.87}$$

$$\|v\|_{0,e}^2 \leq c(h_e^{-1} \|v\|_{0,T}^2 + h_e |v|_{1,T}^2) \quad \forall\, v \in \mathrm{H}^1(T),\ \forall\, T \in \omega_e. \tag{4.88}$$

In addition, using scaling arguments (see [47, Lemma 1.101]) and the continuous embedding $\mathrm{H}^1 \hookrightarrow \mathrm{L}^4$ in the reference triangle, it is not difficult to deduce that

$$\|v\|_{0,4,T} \leq c(h_T^{-1/2} \|v\|_{0,T} + h_T^{1/2} |v|_{1,T}), \quad \forall\, v \in \mathrm{H}^1(T). \tag{4.89}$$

From now on we assume, without loss of generality, that $\theta_h \mathbf{g}_\mathrm{f}$ is polynomial on each $T \in \mathcal{T}_h^\mathrm{f}$, and that $\mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}$ and $\theta_h \mathbf{g}_\mathrm{m}$ are polynomials on each $T \in \mathcal{T}_h^\mathrm{m}$. Otherwise, additional higher order terms, given by the errors arising from suitable polynomial approximations, should appear in the corresponding bounds below, which explains the expression h.o.t. in (4.82).

We continue by providing the following estimate for the discrete equilibrium equation.

Lemma 4.3.19. *For all $T \in \mathcal{T}_h^\mathrm{f}$, there exists $C_1 > 0$, independent of the mesh-size, such that*

$$\begin{aligned} h_T \|\mathbf{div}\, \boldsymbol{\sigma}_{h,\mathrm{f}} - (\mathbf{u}_{h,\mathrm{f}} \cdot \nabla)\mathbf{u}_{h,\mathrm{f}} - \tfrac{1}{2}\mathbf{u}_{h,\mathrm{f}} \mathrm{div}\, \mathbf{u}_{h,\mathrm{f}} + \theta_h \mathbf{g}_\mathrm{f}\|_{0,T} \\ \leq C_1((1 + h_T + h_T^{1/2}) \|\mathbf{u}_\mathrm{f} - \mathbf{u}_{h,\mathrm{f}}\|_{1,T} + \|p - p_h\|_{0,T} + h_T^{1/2} \|\theta - \theta_h\|_{1,T}), \end{aligned}$$

Proof. Given $T \in \mathcal{T}_h^\mathrm{f}$, we define the local polynomial function $\mathbf{X}_T = (\mathbf{div}\, \boldsymbol{\sigma}_{h,\mathrm{f}} - (\mathbf{u}_{h,\mathrm{f}} \cdot \nabla)\mathbf{u}_{h,\mathrm{f}} - \frac{1}{2}\mathbf{u}_{h,\mathrm{f}} \mathrm{div}\, \mathbf{u}_{h,\mathrm{f}} + \theta_h \mathbf{g}_\mathrm{f})|_T$. Then, applying the upper bound (4.84), the fact that $-\mathbf{div}\, \boldsymbol{\sigma}_\mathrm{f} + (\mathbf{u}_\mathrm{f} \cdot \nabla)\mathbf{u}_\mathrm{f} - \theta \mathbf{g}_\mathrm{f} = \mathbf{0}$ and $\mathrm{div}\, \mathbf{u}_\mathrm{f}$ in Ω_f (cf. Theorem 4.3.16), and adding and subtracting suitable terms, we find

$$\begin{aligned} \|\mathbf{X}_T\|_{0,T}^2 &\leq c \|\varphi_T^{1/2} \mathbf{X}_T\|_{0,T}^2 \\ &= c\big[(\varphi_T \mathbf{X}_T, \mathbf{div}\, (\boldsymbol{\sigma}_{h,\mathrm{f}} - \boldsymbol{\sigma}_\mathrm{f}))_T + (\varphi_T \mathbf{X}_T, ((\mathbf{u}_\mathrm{f} - \mathbf{u}_{h,\mathrm{f}}) \cdot \nabla)\mathbf{u}_{h,\mathrm{f}})_T \\ &- (\varphi_T \mathbf{X}_T, (\mathbf{u}_\mathrm{f} \cdot \nabla)(\mathbf{u}_\mathrm{f} - \mathbf{u}_{h,\mathrm{f}}))_T + \tfrac{1}{2}(\varphi_T \mathbf{X}_T, \mathbf{u}_{h,\mathrm{f}} \mathrm{div}\, (\mathbf{u}_\mathrm{f} - \mathbf{u}_{h,\mathrm{f}}))_T \\ &+ (\varphi_T \mathbf{X}_T \cdot (\theta_h - \theta)\mathbf{g}_\mathrm{f})_T\big]. \end{aligned} \tag{4.90}$$

Now, we bound each term on the right-hand side of (4.90).

For the first term we integrate by parts, make use of the fact that $\varphi_T = 0$ on ∂T and $0 \leq \varphi_T \leq 1$, apply the inverse inequality (4.87) with $p = 2$ and $l = 1$, and employ the definitions of $\boldsymbol{\sigma}_{\mathrm{f}}$ and $\boldsymbol{\sigma}_{h,\mathrm{f}}$ (cf. (4.1) and (4.34)), to obtain

$$\begin{aligned}|(\varphi_T \mathbf{X}_T, \mathbf{div}\,(\boldsymbol{\sigma}_{h,\mathrm{f}} - \boldsymbol{\sigma}_{\mathrm{f}}))_T| &= |(\nabla(\varphi_T \mathbf{X}_T), \boldsymbol{\sigma}_{h,\mathrm{f}} - \boldsymbol{\sigma}_{\mathrm{f}})_T| \\ &\leq |\varphi_T \mathbf{X}_T|_{1,T} \|\boldsymbol{\sigma}_{h,\mathrm{f}} - \boldsymbol{\sigma}_{\mathrm{f}}\|_{0,T} \\ &\leq c_1 h_T^{-1} \|\mathbf{X}_T\|_{0,T} (|\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}|_{1,T} + \|p - p_h\|_{0,T}).\end{aligned}$$

In turn, for the second, third and fourth terms we first observe that the inverse inequality (4.87) with $p = 4$ and $l = 0$, and the fact that $0 \leq \varphi_T \leq 1$, imply:

$$\|\varphi_T \mathbf{X}_T\|_{0,4,T} \leq c_2 h_T^{-1/2} \|\mathbf{X}_T\|_{0,T}. \tag{4.91}$$

Then, the Hölder inequality, estimate (4.91), the inverse inequality (4.89), the continuous injection $\mathrm{H}^1(\Omega_{\mathrm{f}}) \hookrightarrow \mathrm{L}^4(\Omega_{\mathrm{f}})$, and the fact that $\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}$ and $\|\mathbf{u}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}$ are bounded, imply

$$\begin{aligned}|(\varphi_T \mathbf{X}_T, ((\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}) \cdot \nabla)\mathbf{u}_{h,\mathrm{f}})_T| &\leq \|\varphi_T \mathbf{X}_T\|_{0,4,T} \|\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}\|_{0,4,T} \|\nabla \mathbf{u}_{h,\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} \\ &\leq c_3 (h_T^{-1} \|\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}\|_{0,T} + |\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}|_{1,T}) \|\mathbf{X}_T\|_{0,T},\end{aligned}$$

$$\begin{aligned}|(\varphi_T \mathbf{X}_T, (\mathbf{u}_{\mathrm{f}} \cdot \nabla)(\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}))_T| &\leq \|\varphi_T \mathbf{X}_T\|_{0,4,T} \|\mathbf{u}_{\mathrm{f}}\|_{0,4,\Omega_{\mathrm{f}}} \|\nabla(\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}})\|_{0,T} \\ &\leq c_4 h_T^{-1/2} |\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}|_{1,T} \|\mathbf{X}_T\|_{0,T},\end{aligned}$$

and

$$\begin{aligned}|(\varphi_T \mathbf{X}_T, \mathbf{u}_{h,\mathrm{f}} \mathrm{div}\,(\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}))_T| &\leq \|\varphi_T \mathbf{X}_T\|_{0,4,T} \|\mathbf{u}_{h,\mathrm{f}}\|_{0,4,\Omega_{\mathrm{f}}} \|\mathrm{div}\,(\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}})\|_{0,T} \\ &\leq c_5 h_T^{-1/2} \|\mathbf{X}_T\|_{0,T} \|\mathrm{div}\,(\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}})\|_{0,T} \\ &\leq c_5 h_T^{-1/2} \|\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}\|_{1,T} \|\mathbf{X}_T\|_{0,T}.\end{aligned}$$

Finally, for the fifth term we use Hölder's inequality, the continuous injection $\mathrm{H}^1(\Omega_{\mathrm{f}}) \hookrightarrow \mathrm{L}^4(\Omega_{\mathrm{f}})$, and estimate (4.91), to obtain

$$\begin{aligned}(\varphi_T \mathbf{X}_T, (\theta_h - \theta)\mathbf{g}_{\mathrm{f}})_T &\leq \|\varphi_T \mathbf{X}_T\|_{0,4,T} \|\mathbf{g}_{\mathrm{f}}\|_{0,4,T} \|\theta_h - \theta\|_{0,T} \\ &\leq c_6 h_T^{-1/2} \|\theta_h - \theta\|_{1,T} \|\mathbf{X}_T\|_{0,T}.\end{aligned}$$

In this way, from the previous estimates we readily deduce the result. □

We continue with the following lemma providing suitable estimates for local quantities in Ω_{f}.

Lemma 4.3.20. *There exist positive constants C_2, C_3, independent of the mesh-size, such that*

(a) $h_e^{1/2}\|[\![\boldsymbol{\sigma}_{h,\mathrm{f}}\mathbf{n}]\!]\|_{0,e} \leq C_2 \sum_{T\in\omega_e}\Big((1+h_T+h_T^{1/2})\|\mathbf{u}_\mathrm{f}-\mathbf{u}_{h,\mathrm{f}}\|_{1,T}+\|p-p_h\|_{0,T}+ h_T^{1/2}\|\theta-\theta_h\|_{0,T}\Big)$, *where* $\omega_e = \cup\{T'\in\mathcal{T}_h^\mathrm{f} : e\in\mathcal{E}(T')\}$.

(b) $h_e^{1/2}\|\boldsymbol{\sigma}_{h,\mathrm{f}}\mathbf{n}+\omega_1(\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{t})\mathbf{t}+\lambda_h\mathbf{n}\|_{0,e} \leq C_3\Big((1+h_{T_e}+h_{T_e}^{1/2})\|\mathbf{u}_\mathrm{f}-\mathbf{u}_{h,\mathrm{f}}\|_{1,T_e}+\|p-p_h\|_{0,T_e}+h_{T_e}^{1/2}\|\theta-\theta_h\|_{0,T_e}+h_e^{1/2}\|\lambda-\lambda_h\|_{0,e}\Big)$, *for all* $e\in\mathcal{E}_h(\Sigma)$, *where* T_e *is the triangle of* $\mathcal{T}_h^\mathrm{f}$ *having* e *as an edge.*

Proof. In what follows we proceed similarly to [4, Lemmas 4.5 and 4.6]. We begin by proving *(a)*. To that end, given $e\in\mathcal{E}_h(\Omega_\mathrm{f})$, we let $\mathbf{X}_e = [\![\boldsymbol{\sigma}_{h,\mathrm{f}}\mathbf{n}]\!]|_e$, employ (4.85), recall that $\varphi_e = 0$ on $\partial\omega_e$, make use of the fact $[\![\boldsymbol{\sigma}_\mathrm{f}\mathbf{n}]\!] = 0$ for all $e\in\mathcal{E}_h(\Omega_\mathrm{f})$ ($\boldsymbol{\sigma}_\mathrm{f}\in\mathbb{H}(\mathbf{div};\Omega_\mathrm{f})$), and integrate by parts, to obtain

$$\begin{aligned}\|\mathbf{X}_e\|_{0,e}^2 &\leq \widehat{c}\|\varphi_e^{1/2}\mathbf{X}_e\|_{0,e}^2 = \widehat{c}\sum_{T\in\omega_e}(\varphi_e\mathbf{L}(\mathbf{X}_e),(\boldsymbol{\sigma}_{h,\mathrm{f}}-\boldsymbol{\sigma}_\mathrm{f})\mathbf{n})_{\partial T}\\ &= \widehat{c}\sum_{T\in\omega_e} -(\varphi_e\mathbf{L}(\mathbf{X}_e),\mathbf{div}\,(\boldsymbol{\sigma}_{h,\mathrm{f}}-\boldsymbol{\sigma}_\mathrm{f}))_T + (\nabla(\varphi_e\mathbf{L}(\mathbf{X}_e)),\boldsymbol{\sigma}_\mathrm{f}-\boldsymbol{\sigma}_{h,\mathrm{f}})_T.\end{aligned}$$

Notice that adding and subtracting suitable terms, and employing the fact that $-\mathbf{div}\,\boldsymbol{\sigma}_\mathrm{f}+(\mathbf{u}_\mathrm{f}\cdot\nabla)\mathbf{u}_\mathrm{f}-\theta\mathbf{g}_\mathrm{f} = \mathbf{0}$ and $\mathrm{div}\,\mathbf{u}_\mathrm{f} = 0$ in Ω_f, the first term above can be rewritten as

$$\begin{aligned}(\varphi_e\mathbf{L}(\mathbf{X}_e),\mathbf{div}\,(\boldsymbol{\sigma}_{h,\mathrm{f}}-\boldsymbol{\sigma}_\mathrm{f}))_T = {}&(\varphi_e\mathbf{L}(\mathbf{X}_e),\mathbf{X}_T)_T\\ &+(\varphi_e\mathbf{L}(\mathbf{X}_e),(\mathbf{u}_{h,\mathrm{f}}\cdot\nabla)(\mathbf{u}_{h,\mathrm{f}}-\mathbf{u}_\mathrm{f}))_T\\ &+(\varphi_e\mathbf{L}(\mathbf{X}_e),((\mathbf{u}_{h,\mathrm{f}}-\mathbf{u}_\mathrm{f})\cdot\nabla)\mathbf{u}_\mathrm{f})_T\\ &+\tfrac{1}{2}(\varphi_e\mathbf{L}(\mathbf{X}_e),\mathbf{u}_{h,\mathrm{f}}\mathrm{div}\,(\mathbf{u}_{h,\mathrm{f}}-\mathbf{u}_\mathrm{f}))_T\\ &+(\varphi_e\mathbf{L}(\mathbf{X}_e),(\theta-\theta_h)\mathbf{g}_\mathrm{f})_T,\end{aligned}\tag{4.92}$$

where $\mathbf{X}_T = (\mathbf{div}\,\boldsymbol{\sigma}_{h,\mathrm{f}}-(\mathbf{u}_{h,\mathrm{f}}\cdot\nabla)\mathbf{u}_{h,\mathrm{f}}-\frac{1}{2}\mathbf{u}_{h,\mathrm{f}}\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}+\theta_h\mathbf{g}_\mathrm{f})|_T$, which implies

$$\begin{aligned}\|\mathbf{X}_e\|_{0,e}^2 \leq \widehat{c}\sum_{T\in\omega_e}\Big\{&(-\varphi_e\mathbf{L}(\mathbf{X}_e),\mathbf{X}_T)_T-(\varphi_e\mathbf{L}(\mathbf{X}_e),(\mathbf{u}_{h,\mathrm{f}}\cdot\nabla)(\mathbf{u}_{h,\mathrm{f}}-\mathbf{u}_\mathrm{f}))_T\\ &-(\varphi_e\mathbf{L}(\mathbf{X}_e),((\mathbf{u}_{h,\mathrm{f}}-\mathbf{u}_\mathrm{f})\cdot\nabla)\mathbf{u}_\mathrm{f})_T-\tfrac{1}{2}(\varphi_e\mathbf{L}(\mathbf{X}_e),\mathbf{u}_{h,\mathrm{f}}\mathrm{div}\,(\mathbf{u}_{h,\mathrm{f}}-\mathbf{u}_\mathrm{f}))_T\\ &-(\varphi_e\mathbf{L}(\mathbf{X}_e),(\theta-\theta_h)\mathbf{g}_\mathrm{f})_T+(\nabla(\varphi_e\mathbf{L}(\mathbf{X}_e)),\boldsymbol{\sigma}_\mathrm{f}-\boldsymbol{\sigma}_{h,\mathrm{f}})_T\Big\}.\end{aligned}\tag{4.93}$$

In turn, analogously as for (4.91) we apply the inverse inequality (4.87) with $p = 4$ and $l = 0$ and the fact that $0 \leq \varphi_e \leq 1$, to deduce that

$$\|\varphi_e \mathbf{L}(\mathbf{X}_e)\|_{0,4,T} \leq c_2 h_T^{-1/2} \|\varphi_e^{1/2} \mathbf{L}(\mathbf{X}_e)\|_{0,T}. \tag{4.94}$$

Then, using Hölder's inequality, estimate (4.94), the inverse inequalities (4.87) and (4.89), and the continuous injection $\mathrm{H}^1(\Omega_\mathrm{f}) \hookrightarrow \mathrm{L}^4(\Omega_\mathrm{f})$, each term on the right-hand side of (4.93) can be bounded as follows

$$\begin{aligned}
|(\varphi_e \mathbf{L}(\mathbf{X}_e), \mathbf{X}_T)_T| &\leq c \|\mathbf{X}_T\|_{0,T} \|\varphi_e^{1/2} \mathbf{L}(\mathbf{X}_e)\|_{0,T}, \\
|(\varphi_e \mathbf{L}(\mathbf{X}_e), (\mathbf{u}_{h,\mathrm{f}} \cdot \nabla)(\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}))_T| &\leq c h_T^{-1/2} \|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} |\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}|_{1,T} \|\varphi_e^{1/2} \mathbf{L}(\mathbf{X}_e)\|_{0,T}, \\
|(\varphi_e \mathbf{L}(\mathbf{X}_e), ((\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}) \cdot \nabla)\mathbf{u}_\mathrm{f})_T| &\leq c \|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} (h_T^{-1} \|\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}\|_{0,T} + |\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}|_{1,T}) \\
&\qquad \times \|\varphi_e^{1/2} \mathbf{L}(\mathbf{X}_e)\|_{0,T}, \\
|(\varphi_e \mathbf{L}(\mathbf{X}_e), \tfrac{1}{2} \mathbf{u}_{h,\mathrm{f}} \mathrm{div}\,(\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}))_T| &\leq c h_T^{-1/2} \|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} |\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}|_{1,T} \|\varphi_e^{1/2} \mathbf{L}(\mathbf{X}_e)\|_{0,T}, \\
|(\varphi_e \mathbf{L}(\mathbf{X}_e), (\theta - \theta_h)\mathbf{g}_\mathrm{f})_T| &\leq c h_T^{-1/2} \|\mathbf{g}_\mathrm{f}\|_{0,4,T} \|\theta_h - \theta\|_{0,T} \|\varphi_e^{1/2} \mathbf{L}(\mathbf{X}_e)\|_{0,T}, \\
|(\nabla(\varphi_e \mathbf{L}(\mathbf{X}_e)), \boldsymbol{\sigma}_\mathrm{f} - \boldsymbol{\sigma}_{h,\mathrm{f}})_T| &\leq h_T^{-1} \|\boldsymbol{\sigma}_\mathrm{f} - \boldsymbol{\sigma}_{h,\mathrm{f}}\|_{0,T} \|\varphi_e^{1/2} \mathbf{L}(\mathbf{X}_e)\|_{0,T}.
\end{aligned}$$

Combining the above with (4.93), employing estimate (4.86), Lemma 4.3.19, the fact that $\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}}$ and $\|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}}$ are bounded, using that $h_e \leq h_T$, and recalling the definition of $\boldsymbol{\sigma}_\mathrm{f}$ and $\boldsymbol{\sigma}_{h,\mathrm{f}}$, we readily deduce *(a)*.

For *(b)*, given $e \in \mathcal{E}_h(\Sigma)$ we let $\mathbf{Y}_e = (\boldsymbol{\sigma}_{h,\mathrm{f}}\mathbf{n} + \omega_1(\mathbf{u}_{h,\mathrm{f}} \cdot \mathbf{t})\mathbf{t} + \lambda_h \mathbf{n})|_e$ recall that $(\boldsymbol{\sigma}_\mathrm{f}\mathbf{n} + \omega_1(\mathbf{u}_\mathrm{f} \cdot \mathbf{t})\mathbf{t} + \lambda \mathbf{n})|_e = 0$ for all $e \in \mathcal{E}_h(\Sigma)$, utilize (4.85) and the fact that $\mathbf{L}(\mathbf{Y}_e)|_e = \mathbf{Y}_e$ and integrate by parts, to deduce that

$$\begin{aligned}
\|\mathbf{Y}_e\|_{0,e}^2 \leq \widetilde{c} \|\varphi_e^{1/2} \mathbf{Y}_e\|_{0,e}^2 &= \widetilde{c} (\varphi_e \mathbf{L}(\mathbf{Y}_e), \boldsymbol{\sigma}_{h,\mathrm{f}}\mathbf{n} + \omega_1(\mathbf{u}_{h,\mathrm{f}} \cdot \mathbf{t})\mathbf{t} + \lambda_h \mathbf{n})_e \\
&= \widetilde{c} (\varphi_e \mathbf{L}(\mathbf{Y}_e), (\boldsymbol{\sigma}_{h,\mathrm{f}} - \boldsymbol{\sigma}_\mathrm{f})\mathbf{n})_e + \omega_1 (\varphi_e \mathbf{L}(\mathbf{Y}_e) \cdot \mathbf{t}, (\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}) \cdot \mathbf{t})_e \\
&\quad + (\varphi_e \mathbf{L}(\mathbf{Y}_e) \cdot \mathbf{n}, \lambda_h - \lambda)_e \\
&= -\widetilde{c} (\varphi_e \mathbf{L}(\mathbf{Y}_e), \mathbf{div}\,(\boldsymbol{\sigma}_{h,\mathrm{f}} - \boldsymbol{\sigma}_\mathrm{f}))_{T_e} + \widetilde{c} (\nabla(\varphi_e \mathbf{L}(\mathbf{Y}_e)), \boldsymbol{\sigma}_\mathrm{f} - \boldsymbol{\sigma}_{h,\mathrm{f}})_{T_e} \\
&\quad + \omega_1 (\varphi_e \mathbf{Y}_e \cdot \mathbf{t}, (\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}) \cdot \mathbf{t})_e + (\varphi_e \mathbf{Y}_e \cdot \mathbf{n}, \lambda_h - \lambda)_e.
\end{aligned}$$

Notice that using (4.88), $\|\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}\|_{0,e}$ can be bound as follows

$$\|\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}\|_{0,e} \leq c \left\{ h_T^{-1/2} \|\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}\|_{0,T_e} + h_T^{1/2} |\mathbf{u}_{h,\mathrm{f}} - \mathbf{u}_\mathrm{f}|_{1,T_e} \right\}.$$

Then, using this estimate and proceeding similarly as for *(a)*, we obtain *(b)*. $\square$

The following lemmas establish some estimates for the local terms involving the temperature.

Lemma 4.3.21. *There exists a positive constant C_4, independent of the mesh-size, such that*

$$\begin{aligned} h_T \|\kappa_{\mathrm{f}} \Delta\theta_h - \mathbf{u}_{h,\mathrm{f}} \cdot \nabla\theta_h - \tfrac{1}{2}\theta_h \mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}\|_{0,T} \; &\leq C_4((1 + h_T + h_T^{1/2})\|\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}\|_{1,T} \\ &+ (1 + h_T^{1/2})|\theta - \theta_h|_{1,T}), \end{aligned}$$

for all $T \in \mathcal{T}_h^{\mathrm{f}}$.

Proof. In what follows, we proceed similarly to [36, Lemma 3.18]. In fact, given $T \in \mathcal{T}_h^{\mathrm{f}}$, we define the local polynomial function $X_T = (\kappa_{\mathrm{f}} \Delta\theta_h - \mathbf{u}_{h,\mathrm{f}} \cdot \nabla\theta_h - \frac{1}{2}\theta_h \mathrm{div}\,\mathbf{u}_{h,\mathrm{f}})|_T$. Then, applying the upper bound (4.84), the fact that $-\kappa_{\mathrm{f}}\,\Delta\theta + \mathbf{u}_{\mathrm{f}} \cdot \nabla\theta = 0$ and $\mathrm{div}\,\mathbf{u}_{\mathrm{f}} = 0$ in Ω_{f} (cf. Theorem 4.3.16), adding and subtracting suitable terms, and integrating by parts, we find that

$$\begin{aligned} \|X_T\|_{0,T}^2 \leq \; & c\|\varphi_T^{1/2} X_T\|_{0,T}^2 \\ = \; & c[\kappa_{\mathrm{f}}(\nabla(\varphi_T X_T), \nabla(\theta - \theta_h))_T + (\varphi_T X_T, \mathbf{u}_{h,\mathrm{f}} \cdot \nabla(\theta - \theta_h))_T \\ + \; & (\varphi_T X_T, (\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}) \cdot \nabla\theta)_T + (\varphi_T X_T, \tfrac{1}{2}\theta_h \mathrm{div}\,(\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}))_T], \end{aligned}$$

which implies

$$\begin{aligned} \|X_T\|_{0,T}^2 \leq \; & c\big(\kappa_{\mathrm{f}}|\varphi_T X_T|_{1,T}|\theta - \theta_h|_{1,T} + \|\varphi_T X_T\|_{0,4,T}\|\mathbf{u}_{h,\mathrm{f}}\|_{0,4,T}|\theta - \theta_h|_{1,T} \\ + \; & \|\varphi_T X_T\|_{0,4,T}\|\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}\|_{0,4,T}|\theta|_{1,T} \\ + \; & \tfrac{\sqrt{2}}{2}\|\varphi_T X_T\|_{0,4,T}\|\theta_h\|_{0,4,T}|\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}|_{1,T}\big), \end{aligned} \tag{4.95}$$

Then, we proceed similarly to the proof of Lemma 4.3.19, by applying estimate (4.91), the inverse inequality (4.89), the continuous injections $\mathrm{H}^1(\Omega_{\mathrm{f}}) \hookrightarrow \mathrm{L}^4(\Omega_{\mathrm{f}})$ and $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$, and the fact that $\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}$, $|\theta|_{1,\Omega}$ and $|\theta_h|_{1,\Omega}$ are bounded, to obtain the result. □

Lemma 4.3.22. *There exists a positive constant C_5, independent of the mesh-size, such that*

$$h_e^{1/2}\|[\![\kappa_{\mathrm{f}}\nabla\theta_h\cdot\mathbf{n}]\!]\|_{0,e} \leq C_5 \sum_{T\in\omega_e} \Big((1+h_T+h_T^{1/2})\|\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}}\|_{1,T}+(1+h_T^{1/2})\|\theta-\theta_h\|_{1,T})\Big),$$

for all $e \in \mathcal{E}_h(\Omega_{\mathrm{f}})$, where ω_e is given by $\omega_e := \cup\{T' \in \mathcal{T}_h^{\mathrm{f}} : e \in \mathcal{E}(T')\}$.

Proof. Given $e \in \mathcal{E}_h(\Omega_{\mathrm{f}})$, similarly to [36, Lemma 3.19] we let $X_e = [\![\kappa_{\mathrm{f}}\nabla\theta_h \cdot \mathbf{n}]\!]|_e$, employ (4.85), recall that $\varphi_e = 0$ on $\partial\omega_e$, make use of the fact that $[\![\kappa_{\mathrm{f}}\nabla\theta\cdot\mathbf{n}]\!]|_e = 0$

for all $e \in \mathcal{E}_h(\Omega_{\rm f})$ and integrate by parts, to deduce that

$$\begin{aligned}\|X_e\|_{0,e}^2 \leq c\|\varphi_e^{1/2} X_e\|_{0,e}^2 &= c\sum_{T\in\omega_e} (\varphi_e L(X_e), \kappa_{\rm f}\nabla\theta_h\cdot\mathbf{n})_{\partial T}\\ &= c\sum_{T\in\omega_e} (\varphi_e L(X_e), \kappa_{\rm f}\nabla(\theta_h-\theta)\cdot\mathbf{n})_{\partial T}\\ &= c\sum_{T\in\omega_e} \kappa_{\rm f}(\nabla(\varphi_e L(X_e)), \nabla(\theta_h-\theta))_T\\ &+ c\sum_{T\in\omega_e} (\varphi_e L(X_e), \Delta(\theta_h-\theta))_T.\end{aligned}$$

Then, adding and subtracting suitable terms, and using the fact that $-\kappa_{\rm f}\Delta\theta + \mathbf{u}_{\rm f}\cdot\nabla\theta = 0$ and $\mathrm{div}\,\mathbf{u}_{\rm f} = 0$ in $\Omega_{\rm f}$, the previous estimate implies

$$\begin{aligned}\|X_e\|_{0,e}^2 \;&\leq c\sum_{T\in\omega_e}\Big((\varphi_e L(X_e), X_T)_T + (\nabla(\varphi_e L(X_e)), \kappa_{\rm f}\nabla(\theta_h-\theta))_T\\ &+ (\varphi_e L(X_e), \tfrac{1}{2}\theta_h \mathrm{div}\,(\mathbf{u}_{h,\rm f}-\mathbf{u}_{\rm f}))_T\\ &+ (\varphi_e L(X_e), (\mathbf{u}_{h,\rm f}-\mathbf{u}_{\rm f})\cdot\nabla\theta)_T\\ &+ (\varphi_e L(X_e), \mathbf{u}_{h,\rm f}\cdot\nabla(\theta_h-\theta))_T\Big),\end{aligned} \tag{4.96}$$

where $X_T = (\kappa_{\rm f}\Delta\theta_h - \mathbf{u}_{h,\rm f}\cdot\nabla\theta_h - \frac{1}{2}\theta_h\mathrm{div}\,\mathbf{u}_{h,\rm f})|_T$. Then, proceeding similarly as in Lemma 4.3.20 (a), we bound each term in the right-hand side of (4.96) as follows

$$\begin{aligned}|(\varphi_e L(X_e), X_T)_T| &\leq c\|\varphi_e^{1/2}L(X_e)\|_{0,T}\|X_T\|_{0,T},\\ |(\nabla(\varphi_e L(X_e)), \kappa_{\rm f}\nabla(\theta_h-\theta))_T| &\leq c h_T^{-1}\|\varphi_e^{1/2}L(X_e)\|_{0,T}|\theta_h-\theta|_{1,T},\\ |(\varphi_e L(X_e), \tfrac{1}{2}\theta_h\mathrm{div}\,(\mathbf{u}_{h,\rm f}-\mathbf{u}_{\rm f}))_T| &\leq c h_T^{-1/2}\|\varphi_e^{1/2}L(X_e)\|_{0,T}\|\theta_h\|_{1,\Omega}|\mathbf{u}_{h,\rm f}-\mathbf{u}_{\rm f}|_{1,T},\\ |(\varphi_e L(X_e), (\mathbf{u}_{h,\rm f}-\mathbf{u}_{\rm f})\cdot\nabla\theta)_T| &\leq c\|\varphi_e^{1/2}L(X_e)\|_{0,T}|\theta|_{1,\Omega}\\ &\quad\times(h_T^{-1}\|\mathbf{u}_{h,\rm f}-\mathbf{u}_{\rm f}\|_{0,T} + |\mathbf{u}_{h,\rm f}-\mathbf{u}_{\rm f}|_{1,T}),\\ |(\varphi_e L(X_e), \mathbf{u}_{h,\rm f}\cdot\nabla(\theta_h-\theta))_T| &\leq c h_T^{-1/2}\|\varphi_e^{1/2}L(X_e)\|_{0,T}\|\mathbf{u}_{h,\rm f}\|_{1,\Omega_{\rm f}}|\theta-\theta_h|_{1,T}.\end{aligned}$$

Then, using Lemma 4.3.21, the upper bound $h_e \leq h_T$, estimate (4.86) and the fact that $\|\theta_h\|_{1,\Omega}$ and $\|\mathbf{u}_{h,\rm f}\|_{1,\Omega_{\rm f}}$ are bounded, we achieve the result. □

Lemma 4.3.23. *There exists a positive constant C_6, independent of the mesh-size, such that*

$$h_T\|\kappa_{\rm m}\Delta\theta_h - \mathbf{u}_{h,\rm m}\cdot\nabla\theta_h\|_{0,T} \leq C_6(\|\theta-\theta_h\|_{1,T} + h_T^{1/2}\|\mathbf{u}_{\rm m}-\mathbf{u}_{h,\rm m}\|_{0,T}),$$

for all $T \in \mathcal{T}_h^{\rm m}$.

Proof. Proceeding analogously to the proof of Lemma 4.3.21, for a given $T \in \mathcal{T}_h^{\mathrm{m}}$, we define the local polynomial function $X_T = (\kappa_{\mathrm{m}}\Delta\theta_h - \mathbf{u}_{h,\mathrm{m}} \cdot \nabla\theta_h)|_T$. Using the upper bound (4.84), the fact that $-\kappa_{\mathrm{m}}\Delta\theta + \mathbf{u}_{\mathrm{m}} \cdot \nabla\theta = 0$ in Ω_{m} (cf. Theorem 4.3.16), adding and subtracting suitable terms, and integrating by parts, we deduce:

$$
\begin{aligned}
\|X_T\|_{0,T}^2 \le c\|\varphi_T^{1/2} X_T\|_{0,T}^2 &= c[\kappa_{\mathrm{m}}(\nabla(\varphi_T X_T), \nabla(\theta - \theta_h))_T \\
&+ (\varphi_T X_T, \mathbf{u}_{h,\mathrm{m}} \cdot \nabla(\theta - \theta_h))_T \\
&+ (\varphi_T X_T, (\mathbf{u}_{\mathrm{m}} - \mathbf{u}_{h,\mathrm{m}}) \cdot \nabla\theta)_T].
\end{aligned} \tag{4.97}
$$

Using Hölder's inequality, estimate (4.91), and the inverse inequality (4.87), we bound each term on the right-hand side of (4.97) as follows:

$$
|\kappa_{\mathrm{m}}(\nabla(\varphi_T X_T), \nabla(\theta - \theta_h))_T| \le c h_T^{-1} |\theta - \theta_h|_{1,T} \|X_T\|_{0,T},
$$

$$
\begin{aligned}
|(\varphi_T X_T, \mathbf{u}_{h,\mathrm{m}} \cdot \nabla(\theta - \theta_h))_T| &\le \|\varphi_T X_T\|_{0,4,T} \|\mathbf{u}_{h,\mathrm{m}}\|_{0,4,T} |\theta - \theta_h|_{1,T} \\
&\le c h_T^{-1} \|X_T\|_{0,T} \|\mathbf{u}_{h,\mathrm{m}}\|_{0,T} |\theta - \theta_h|_{1,T},
\end{aligned}
$$

$$
|(\varphi_T X_T, (\mathbf{u}_{\mathrm{m}} - \mathbf{u}_{h,\mathrm{m}}) \cdot \nabla\theta)_T| \le c h_T^{-1/2} \|\mathbf{u}_{\mathrm{m}} - \mathbf{u}_{h,\mathrm{m}}\|_{0,T} \|\theta\|_{1,4,\Omega_{\mathrm{m}}} \|X_T\|_{0,T}.
$$

Then, combining these bounds with (4.97), and using the facts that $\|\mathbf{u}_{h,\mathrm{m}}\|_{0,T} \le \|\mathbf{u}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}}$ and $\|\theta\|_{1,4,\Omega_{\mathrm{m}}}$ are bounded, we readily obtain the result. $\square$

Lemma 4.3.24. *There exists a positive constant C_7, independent of the mesh-size, such that*

$$
h_e^{1/2} \|[\![\kappa_{\mathrm{m}} \nabla\theta_h \cdot \mathbf{n}]\!]\|_{0,e} \le C_7 \sum_{T \in \omega_e} \left(\|\theta - \theta_h\|_{1,T} + h_T^{1/2} \|\mathbf{u}_{\mathrm{m}} - \mathbf{u}_{h,\mathrm{m}}\|_{0,T} \right),
$$

for all $e \in \mathcal{E}_h(\Omega_{\mathrm{m}})$, where ω_e is given by $\omega_e := \cup\{T' \in \mathcal{T}_h^{\mathrm{m}} : e \in \mathcal{E}(T')\}$.

Proof. Proceeding analogously to the proof of Lemma 4.3.22. for given $e \in \mathcal{E}_h(\Omega_{\mathrm{m}})$, we let $X_e = [\![\kappa_{\mathrm{m}} \nabla\theta_h \cdot \mathbf{n}]\!]|_e$, employ (4.85), recall that $\varphi_e = 0$ on $\partial\omega_e$ and $[\![\kappa_{\mathrm{m}} \nabla\theta \cdot \mathbf{n}]\!]|_e = 0$ for all $e \in \mathcal{E}_h(\Omega_{\mathrm{m}})$, integrate by parts, add and subtract suitable terms, and use the fact that $-\kappa_{\mathrm{m}}\Delta\theta + \mathbf{u}_{\mathrm{m}} \cdot \nabla\theta = 0$ in Ω_{m} to obtain

$$
\begin{aligned}
\|X_e\|_{0,e}^2 \le\ & c\|\varphi_e^{1/2} X_e\|_{0,e}^2 \\
=\ & c \sum_{T \in \omega_e} (\varphi_e L(X_e), \kappa_{\mathrm{m}} \nabla(\theta_h - \theta) \cdot \mathbf{n})_{\partial T} \\
=\ & c \sum_{T \in \omega_e} \kappa_{\mathrm{m}}(\nabla(\varphi_e L(X_e)), \nabla(\theta_h - \theta))_T + (\varphi_e L(X_e), \Delta(\theta_h - \theta))_T \\
\le\ & c \sum_{T \in \omega_e} \Big((\varphi_e L(X_e), X_T)_T + (\nabla(\varphi_e L(X_e)), \kappa_{\mathrm{m}} \nabla(\theta_h - \theta))_T \\
+\ & (\varphi_e L(X_e), (\mathbf{u}_{h,\mathrm{m}} - \mathbf{u}_{\mathrm{m}}) \cdot \nabla\theta)_T + (\varphi_e L(X_e), \mathbf{u}_{h,\mathrm{m}} \cdot \nabla(\theta_h - \theta))_T \Big),
\end{aligned} \tag{4.98}
$$

where $X_T = (\kappa_{\rm m}\Delta\theta_h - \mathbf{u}_{h,\rm m}\cdot\nabla\theta_h)|_T$.

Now we observe that, similarly to the proof of Lemma 4.3.23, each term on the right-hand side of (4.98) as follows

$$
\begin{aligned}
|(\varphi_e L(X_e), X_T)_T| &\le c\|\varphi_e^{1/2} L(X_e)\|_{0,T}\|X_T\|_{0,T},\\
|(\nabla(\varphi_e L(X_e)), \kappa_{\rm m}\nabla(\theta_h-\theta))_T| &\le c h_T^{-1}\|\varphi_e^{1/2} L(X_e)\|_{0,T}|\theta_h-\theta|_{1,T},\\
|(\varphi_e L(X_e), (\mathbf{u}_{h,\rm m}-\mathbf{u}_{\rm m})\cdot\nabla\theta)_T| &\le c h_T^{-1/2}\|\varphi_e^{1/2} L(X_e)\|_{0,T}\|\mathbf{u}_{h,\rm m}-\mathbf{u}_{\rm m}\|_{0,T}\|\theta\|_{1,4,\Omega_{\rm m}},\\
|(\varphi_e L(X_e), \mathbf{u}_{h,\rm m}\cdot\nabla(\theta_h-\theta))_T| &\le c h_T^{-1}\|\varphi_e^{1/2} L(X_e)\|_{0,T}\|\mathbf{u}_{h,\rm m}\|_{0,T}|\theta-\theta_h|_{1,T},
\end{aligned}
$$

which combined with (4.98), estimate (4.86), Lemma 4.3.23, the upper bound $h_e \le h_T$ and the facts that $\|\mathbf{u}_{h,\rm m}\|_{0,T} \le \|\mathbf{u}_{h,\rm m}\|_{{\rm div};\Omega_{\rm m}}$ and $\|\theta\|_{1,4,\Omega_{\rm m}}$ are bounded, we deduce the result. □

Lemma 4.3.25. *There exists a positive constant C_8, independent of the mesh-size, such that*

$$
\begin{aligned}
h_e^{1/2}\|\kappa_{\rm f}\nabla\theta_{h,\rm f}\cdot\mathbf{n}-\kappa_{\rm m}\nabla\theta_{h,\rm m}\cdot\mathbf{n}\|_{0,e} \le C_8\Big(&(1+h_{T_{\rm f}}+h_{T_{\rm f}}^{1/2})\|\mathbf{u}_{\rm f}-\mathbf{u}_{h,\rm f}\|_{1,T_{\rm f}}\\
&+(1+h_{T_{\rm f}}^{1/2})\|\theta-\theta_h\|_{1,T_{\rm f}}\\
&+\|\theta-\theta_h\|_{1,T_{\rm m}}+h_{T_{\rm m}}^{1/2}\|\mathbf{u}_{\rm m}-\mathbf{u}_{h,\rm m}\|_{0,T_{\rm m}}\Big),
\end{aligned}
$$

for all $e\in\mathcal{E}_h(\Sigma)$, where $T_{\rm f}$ and $T_{\rm m}$ are the triangles of $\mathcal{T}_h^{\rm f}$ and $\mathcal{T}_h^{\rm m}$, respectively, having e as an edge.

Proof. Given $e\in\mathcal{E}_h(\Sigma)$, we let $T_{\rm f}$ and $T_{\rm m}$ be the triangles of $\mathcal{T}_h^{\rm f}$ and $\mathcal{T}_h^{\rm m}$, respectively, having e as an edge, and define $\mathrm{X}_e := (\kappa_{\rm f}\nabla\theta_{h,\rm f}\cdot\mathbf{n}-\kappa_{\rm m}\nabla\theta_{h,\rm m}\cdot\mathbf{n})|_e$ on e. Then, applying (4.85), using that $\kappa_{\rm f}\nabla\theta_{\rm f}\cdot\mathbf{n}-\kappa_{\rm m}\nabla\theta_{\rm m}\cdot\mathbf{n}=0$ on Σ, recalling that $\varphi_e=0$ on $\partial T_{\rm m}\setminus e$ and on $\partial T_{\rm f}\setminus e$, and integrating by parts on $T_{\rm f}$ and $T_{\rm m}$, we get

$$
\begin{aligned}
\|X_e\|_{0,e}^2 &\le c\|\varphi_e^{1/2}X_e\|_{0,e}^2\\
&= c(\varphi_e X_e, \kappa_{\rm f}\nabla(\theta_{h,\rm f}-\theta_{\rm f})\cdot\mathbf{n})_e + c(\varphi_e X_e, \kappa_{\rm m}\nabla(\theta_{\rm m}-\theta_{h,\rm m})\cdot\mathbf{n})_e\\
&= c(\nabla(L(\varphi_e X_e)), \kappa_{\rm f}\nabla(\theta_{h,\rm f}-\theta_{\rm f}))_{T_{\rm f}} + c(\nabla(L(\varphi_e X_e)), \kappa_{\rm m}\nabla(\theta_{\rm m}-\theta_{h,\rm m}))_{T_{\rm m}}\\
&\quad + c(L(\varphi_e X_e), \kappa_{\rm f}\Delta(\theta_{h,\rm f}-\theta_{\rm f}))_{T_{\rm f}} + c(L(\varphi_e X_e), \kappa_{\rm m}\Delta(\theta_{\rm m}-\theta_{h,\rm m}))_{T_{\rm m}}.
\end{aligned}
\tag{4.99}
$$

For the third and fourth terms in (4.99) we add and subtract suitable terms and use the facts that $-\kappa_{\rm f}\,\Delta\theta+\mathbf{u}_{\rm f}\cdot\nabla\theta=0$ in $\Omega_{\rm f}$, ${\rm div}\,\mathbf{u}_{\rm f}=0$ in $\Omega_{\rm f}$ and $-\kappa_{\rm m}\Delta\theta+\mathbf{u}_{\rm m}\cdot\nabla\theta=0$

in Ω_{m} (cf. Theorem 4.3.16), to obtain

$$\begin{aligned}(L(\varphi_e X_e),\kappa_{\mathrm{f}}\Delta(\theta_{h,\mathrm{f}}-\theta_{\mathrm{f}}))_{T_{\mathrm{f}}} &= (L(\varphi_e X_e),X_{T,\mathrm{f}})_{T_{\mathrm{f}}}\\ &\quad + (L(\varphi_e X_e),(\mathbf{u}_{h,\mathrm{f}}-\mathbf{u}_{\mathrm{f}})\cdot\nabla\theta_h)_{T_{\mathrm{f}}}\\ &\quad + (L(\varphi_e X_e),\mathbf{u}_{\mathrm{f}}\cdot\nabla(\theta_h-\theta))_{T_{\mathrm{f}}}\\ &\quad + (L(\varphi_e X_e),\tfrac{1}{2}\mathbf{u}_{h,\mathrm{f}}\mathrm{div}\,(\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}}))_{T_{\mathrm{f}}},\end{aligned}\tag{4.100}$$

and

$$\begin{aligned}(L(\varphi_e X_e),\kappa_{\mathrm{m}}\Delta(\theta_{h,\mathrm{m}}-\theta_{\mathrm{m}}))_{T_{\mathrm{m}}} &= (L(\varphi_e X_e),X_{T,\mathrm{m}})_{T_{\mathrm{m}}}\\ &\quad + (L(\varphi_e X_e),(\mathbf{u}_{h,\mathrm{m}}-\mathbf{u}_{\mathrm{m}})\cdot\nabla\theta_h)_{T_{\mathrm{m}}}\\ &\quad + (L(\varphi_e X_e),\mathbf{u}_{\mathrm{m}}\cdot\nabla(\theta_h-\theta))_{T_{\mathrm{m}}}.\end{aligned}\tag{4.101}$$

with $X_{T,\mathrm{f}} := \kappa_{\mathrm{f}}\Delta\theta_{h,\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}\cdot\nabla\theta_h - \frac{1}{2}\mathbf{u}_{h,\mathrm{f}}\mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}$ and $X_{T,\mathrm{m}} := \kappa_{\mathrm{m}}\Delta\theta_{h,\mathrm{m}} - \mathbf{u}_{h,\mathrm{m}}\cdot\nabla\theta_h$. Then, noticing that each term on the right-hand side of (4.100) and (4.101) can be bound as in Lemmas 4.3.22 and 4.3.24, respectively, and observing that from (4.86) and (4.87), we have

$$|(\nabla(L(\varphi_e X_e)),\kappa_{\mathrm{f}}\nabla(\theta_{h,\mathrm{f}}-\theta_{\mathrm{f}}))_{T_{\mathrm{f}}}| \leq c h_{T_{\mathrm{f}}}^{-1/2}\|X_e\|_{0,T_{\mathrm{f}}}|\theta_{h,\mathrm{f}}-\theta_{\mathrm{f}}|_{1,T_{\mathrm{f}}},$$

and

$$|(\nabla(L(\varphi_e X_e)),\kappa_{\mathrm{m}}\nabla(\theta_{\mathrm{m}}-\theta_{h,\mathrm{m}}))_{T_{\mathrm{m}}}| \leq c h_{T_{\mathrm{m}}}^{-1/2}\|X_e\|_{0,T_{\mathrm{m}}}|\theta_{h,\mathrm{m}}-\theta_{\mathrm{m}}|_{1,T_{\mathrm{m}}},$$

we readily conclude the proof. □

In the following lemma, we summarize known estimates that are either well-established in the literature or can be readily derived from related results.

Lemma 4.3.26. *There exists a positive constants C_i, $i\in\{9,\ldots,14\}$, independent of the mesh-size, such that*

(c) $h_T^2\|\mathrm{rot}\,(\theta_h\mathbf{g}_{\mathrm{m}} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}})\|_{0,T}^2 \leq C_9(\|\mathbf{u}_{\mathrm{m}}-\mathbf{u}_{h,\mathrm{m}}\|_{0,T}^2 + \|\theta-\theta_h\|_{1,T}^2)$, *for all* $T\in\mathcal{T}_h^{\mathrm{m}}$,

(d) $h_e\|[\![(\theta_h\mathbf{g}_{\mathrm{m}} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}})\cdot\mathbf{t}]\!]\|_{0,e}^2 \leq C_{10}(\|\mathbf{u}_{\mathrm{m}}-\mathbf{u}_{h,\mathrm{m}}\|_{0,\omega_e}^2 + \|\theta-\theta_h\|_{1,\omega_e}^2)$, *for all* $e\in\mathcal{E}_h(\Omega_{\mathrm{m}})$, *where the set* ω_e *is given by* $\omega_e := \cup\{T'\in\mathcal{T}_h^{\mathrm{m}} : e\in\mathcal{E}(T')\}$,

(e) $h_T^2\|\theta_h\mathbf{g}_{\mathrm{m}} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}\|_{0,T}^2 \leq C_{11}(h_T^2\|\mathbf{u}_{\mathrm{m}}-\mathbf{u}_{h,\mathrm{m}}\|_{0,T}^2 + h_T^2\|\theta-\theta_h\|_{1,T}^2 + \|p_{\mathrm{m}} - p_{h,\mathrm{m}}\|_{0,T}^2)$, *for all* $T\in\mathcal{T}_h^{\mathrm{m}}$,

(f) $h_e\|\mathbf{u}_{h,\mathrm{f}}\cdot\mathbf{n} - \mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{n}\|_{0,e}^2 \leq C_{12}(\|\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}}\|_{0,T_{\mathrm{f}}}^2 + h_{T_{\mathrm{f}}}^2|\mathbf{u}_{\mathrm{f}}-\mathbf{u}_{h,\mathrm{f}}|_{1,T_{\mathrm{f}}}^2 + \|\mathbf{u}_{\mathrm{m}} - \mathbf{u}_{h,\mathrm{m}}\|_{0,T_{\mathrm{m}}}^2)$, *for all* $e\in\mathcal{E}_h(\Sigma)$, *where* T_{f} *and* T_{m} *are the triangles of* $\mathcal{T}_h^{\mathrm{f}}$ *and* $\mathcal{T}_h^{\mathrm{m}}$, *respectively, having* e *as an edge,*

(g) $h_e\|p_{h,\mathrm{m}} - \lambda_h\|_{0,e}^2 \le C_{13}(h_T^2\|\mathbf{u}_\mathrm{m} - \mathbf{u}_{h,\mathrm{m}}\|_{0,T_e}^2 + h_T^2\|\theta - \theta_h\|_{0,T_e}^2 + \|p_\mathrm{m} - p_{h,\mathrm{m}}\|_{0,T_e}^2 + h_e\|\lambda - \lambda_h\|_{0,e}^2)$, *for all* $e \in \mathcal{E}_h(\Sigma)$, *where* T_e *is the triangle of* $\mathcal{T}_h^\mathrm{m}$ *having* e *as an edge,*

(h) $\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\theta_h\mathbf{g}_\mathrm{m}\cdot\mathbf{t} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{t} - \frac{d\lambda_h}{d\mathbf{t}}\|_{0,e}^2 \le C_{14}(\sum_{e\in\mathcal{E}_h(\Sigma)}(\|\mathbf{u}_\mathrm{m} - \mathbf{u}_{h,\mathrm{m}}\|_{0,T_e}^2 + \|\theta - \theta_h\|_{0,T_e}^2) + \|\lambda - \lambda_h\|_{1/2,\Sigma}^2)$, *where given* $e \in \mathcal{E}_h(\Sigma)$, T_e *is the triangle of* $\mathcal{T}_h^\mathrm{m}$ *having* e *as an edge.*

Proof. The estimates *(c)*, *(d)*, and *(e)* follow by proceeding analogously to [23, Lemma 6.1, 6.2, and 6.3], while for (f) and (g) we refer to [4, Lemma 4.7 and 4.12], respectively. Finally, estimate (h) result from very slight modifications of the proof of [51, Lemma 5.7]. □

We observe here that (h) is the only non-local efficiency bound obtained so far. However, the following lemma shows that a local estimate can still be derived for this term under an additional regularity assumption on λ.

Lemma 4.3.27. *Assume that* $\lambda|_e \in \mathrm{H}^1(e)$ *for each* $e \in \mathcal{E}_h(\Sigma)$. *There exist* $\widetilde{C}_{12} > 0$, *such that*

$$h_e\|\theta_h\mathbf{g}_\mathrm{m}\cdot\mathbf{t} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}\cdot\mathbf{t} - \frac{d\lambda_h}{d\mathbf{t}}\|_{0,e}^2 \le \widetilde{C}_{14}(\|\mathbf{u}_\mathrm{m} - \mathbf{u}_{h,\mathrm{m}}\|_{0,T_e}^2 + \|\theta - \theta_h\|_{0,T_e}^2 + h_e\|\frac{d\lambda}{d\mathbf{t}} - \frac{d\lambda_h}{d\mathbf{t}}\|_{0,e}^2),$$

for all $e \in \mathcal{E}_h(\Sigma)$, *and where* T_e *is the triangle of* $\mathcal{T}_h^\mathrm{m}$ *having* e *as and edge.*

Proof. Similarly as for (h) from Lemma 4.3.26, it follows by adapting the corresponding elasticity version from [54, Lemma 21]. We omit the details. □

We end this section by observing that the efficiency estimate (4.82) follows straightforwardly from Lemmas 4.3.17–4.3.27. In particular, the term $h_e\|\lambda - \lambda_h\|_{0,e}$, which appears in Lemma 4.3.20 (item (b)), is bounded as follows:

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\lambda - \lambda_h\|_{0,e}^2 \le h\|\lambda - \lambda_h\|_{0,\Sigma}^2 \le ch\|\lambda - \lambda_h\|_{1/2,\Sigma}^2.$$

4.4 A posteriori error analysis to the three-dimensional case

In this section we briefly discuss how the a posteriori error analysis can be extended to the three dimensional case. To that end, we first need to introduce some additional notations. Given a tetrahedron $T \in \mathcal{T}_h = \mathcal{T}_h^\mathrm{f} \cup \mathcal{T}_h^\mathrm{m}$, we let $\mathcal{E}_h(T)$ denote the set of faces of T. Furthermore, we define

$$\mathcal{E}_h := \mathcal{E}_h(\Omega_\mathrm{f}) \cup \mathcal{E}_h(\Omega_\mathrm{m}) \cup \mathcal{E}_h(\Gamma_\mathrm{f}) \cup \mathcal{E}_h(\Gamma_\mathrm{m}) \cup \mathcal{E}_h(\Sigma),$$

as the set of all faces of $\mathcal{T}_h$. Here, $\mathcal{E}_h(\Omega_\star)$ and $\mathcal{E}_h(\Gamma_\star)$ are the sets of all internal faces and boundary faces of $\mathcal{T}_h^\star$, respectively, with $\star \in \{\mathrm{f}, \mathrm{m}\}$. The interface faces $\mathcal{E}_h(\Sigma)$ are those forming the previously defined partition Σ_h.

In turn, for any sufficiently smooth vector field $\mathbf{v} = (v_1, v_2, v_3)$, we let

$$\operatorname{curl}(\mathbf{v}) = \nabla \times \mathbf{v} := \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right).$$

Then, the local error indicators $\eta_{T,\mathrm{f}}$, $\eta_{T,\mathrm{m}}$ and $\eta_{T,\Sigma}$ now read

$$\begin{aligned}
\eta_{T,\mathrm{f}}^2 := \;& \|\mathbf{div}\,\mathbf{u}_{h,\mathrm{f}}\|_{0,T}^2 + h_T^2 \|\mathbf{div}\,\boldsymbol{\sigma}_{h,\mathrm{f}} - (\mathbf{u}_{h,\mathrm{f}} \cdot \nabla)\mathbf{u}_{h,\mathrm{f}} - \tfrac{1}{2}\mathbf{u}_{h,\mathrm{f}} \mathrm{div}\,\mathbf{u}_{h,\mathrm{f}} + \theta_h \mathbf{g}_\mathrm{f}\|_{0,T}^2 \\
& + h_T^2 \|\kappa_\mathrm{f} \Delta\theta_h - \mathbf{u}_{h,\mathrm{f}} \cdot \nabla\theta_h - \tfrac{1}{2}\theta_h \mathrm{div}\,\mathbf{u}_{h,\mathrm{f}}\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_\mathrm{f})} h_e \|[\![\boldsymbol{\sigma}_{h,\mathrm{f}} \mathbf{n}]\!]\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_\mathrm{f})} h_e \|[\![\kappa_\mathrm{f} \nabla\theta_h]\!]\|_{0,e}^2,
\end{aligned}$$

and

$$\begin{aligned}
\eta_{T,\mathrm{m}}^2 := \;& h_T^2 \|\operatorname{curl}(\theta_h \mathbf{g}_\mathrm{m} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}})\|_{0,T}^2 + h_T^2 \|\theta_h \mathbf{g}_\mathrm{m} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}\|_{0,T}^2 \\
& + h_T^2 \|\kappa_\mathrm{m} \Delta\theta_h - \mathbf{u}_{h,\mathrm{m}} \cdot \nabla\theta_h\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_\mathrm{m})} h_e \|[\![(\theta_h \mathbf{g}_\mathrm{m} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}) \times \mathbf{n}]\!]\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_\mathrm{m})} h_e \|[\![\kappa_\mathrm{m} \nabla\theta_h]\!]\|_{0,e}^2,
\end{aligned}$$

$$\begin{aligned}
\eta_{T,\Sigma}^2 := & \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \|\kappa_\mathrm{f} \nabla\theta_{h,\mathrm{f}} \cdot \mathbf{n} - \kappa_\mathrm{m} \nabla\theta_{h,\mathrm{m}} \cdot \mathbf{n}\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \|\mathbf{u}_{h,\mathrm{f}} \cdot \mathbf{n} - \mathbf{u}_{h,\mathrm{m}} \cdot \mathbf{n}\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \|(\theta_h \mathbf{g}_\mathrm{m} - \mathbf{K}^{-1}\mathbf{u}_{h,\mathrm{m}}) \times \mathbf{n} - \nabla\lambda_h \times \mathbf{n}\|_{0,e}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \|\lambda_h - p_h\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Sigma)} h_e \|\boldsymbol{\sigma}_{h,\mathrm{f}} \mathbf{n} + \sum_{j=1}^{2} \omega_j (\mathbf{u}_{h,\mathrm{f}} \cdot \mathbf{t}_j)\mathbf{t}_j + \lambda_h \mathbf{n}\|_{0,e}^2,
\end{aligned}$$

and the global a posteriori error indicator is defined as

$$\boldsymbol{\eta} := \left\{ \sum_{T \in \mathcal{T}_h^\mathrm{f}} \eta_{T,\mathrm{f}}^2 + \sum_{T \in \mathcal{T}_h^\mathrm{m}} \eta_{T,\mathrm{m}}^2 + \sum_{T \in \mathcal{T}_h^\Sigma} \eta_{T,\Sigma}^2 \right\}^{1/2},$$

where $\mathcal{T}_h^\Sigma$ is the set of polyhedral of $\mathcal{T}_h$ with faces on the interface Σ.

The reliability of this estimator can be proved essentially by using the same arguments employed for the two dimensional case. In particular, analogously to the 2D case, here it is needed a stable Helmholtz decomposition for $\mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}})$. This result taken from [53, Theorem 3.2] is established next.

Lemma 4.4.1. *Assume that there exists a convex domain Ξ such that $\Omega_{\mathrm{m}} \subseteq \Xi$ and $\Gamma_{\mathrm{m}} \subseteq \partial\Xi$. Then, given $\mathbf{v}_{\mathrm{m}} \in \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}})$, there exists $z \in \mathrm{H}^2(\Omega_{\mathrm{m}})$ and $\boldsymbol{\beta} \in \mathbf{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})$ such that*

$$\mathbf{v}_{\mathrm{m}} = \nabla z + \mathrm{curl}\,(\boldsymbol{\beta}) \quad \textit{in} \quad \Omega_{\mathrm{m}} \quad \textit{and} \quad \|z\|_{2,\Omega_{\mathrm{m}}} + \|\boldsymbol{\beta}\|_{1,\Omega_{\mathrm{m}}} \leq c\|\mathbf{v}_{\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}},$$

where c is a positive constant independent of all the foregoing variables.

On the other hand, the efficiency of the estimator can be derived exactly as for the 2D case, using standard results based on inverse inequalities and localization techniques based on tetrahedron-bubble and face-bubble functions (see for instance [24, Section 3.4] and [43, Section 4.2]).

Chapter 5

A mass conservative finite element method for a nonisothermal fluid-membrane interaction

5.1 Introduction

In this chapter, we proceed similarly to [21] and [70], and consider an $\mathbf{H}(\mathrm{div})$-conforming approximation for the velocity in both, the free-fluid region and the porous medium domain, to propose a strongly mass conservative discretization for the nonisothermal fluid-membrane model introduced in Chapter 2. With the above approach, and unlike in Chapter 3, the continuity of the normal components of the velocity is exactly preserved along the interface, eliminating the need to introduce the Lagrange multiplier. Thus, the resulting unknowns of the problem are the velocity, pressure and temperature in both domains. To examine the well-posedness of the resulting variational formulation, we introduce a reduced problem that is equivalent to the one presented in Chapter 3. Consequently, the existence and uniqueness of the solution are established using the same techniques.

Additionally, the method has the distinct advantage of yielding exactly divergence-free velocity approximations, allowing for the analysis of the discrete scheme without modifying the convective term of the heat equation. For the discretization, we use Brezzi–Douglas–Marini (BDM) elements of order k for the velocities, discontinuous elements of order $k-1$ for the pressure, and standard continuous elements of order k for the temperature in both the free-fluid and porous media domains, while H^1 continuity of the velocity in the free-fluid region is enforced through an interior penalty discontinuous Galerkin (DG) technique as in [34].

The analysis of the discrete problem is performed using a sufficiently small data assumption and a fixed-point strategy. Additionally, by imposing a smallness condition on the data and the temperature in the membrane, we establish the convergence of the Galerkin scheme and derive the corresponding theoretical rate of convergence. To validate these theoretical results, we conduct several numerical simulations based on manufactured solutions.

The remainder of the chapter is organized as follows.

- In **Section 5.2**, we derive the corresponding variational formulation and analyze the existence and uniqueness of the solution.

- In **Section 5.3**, we introduce the mass-conservative numerical scheme and analyze its well-posedness.

- In **Section 5.4**, we develop the error analysis and derive the corresponding order of convergence of the scheme.

- In **Section 5.5**, we provide some numerical results illustrating the performance of the method and confirming the theoretical rate of convergence.

5.2 Continuous problem

In this section, we present an alternative variational formulation established in Chapter 3 for the model problem given in Chapter 2. In what follows, we derive the weak formulation and we establish the existence and uniqueness of solution.

5.2.1 The model problem

In this section, we briefly recall the modeling problem described in Chapter 2. Let Ω_{f} and Ω_{m} be two bounded and simply connected open polygonal or polyhedral domains in $\mathbb{R}^d$, with $d = 2$ or $d = 3$, respectively, such that $\Omega_{\mathrm{f}} \cap \Omega_{\mathrm{m}} = \emptyset$ and $\Sigma := \partial\Omega_{\mathrm{f}} \cap \partial\Omega_{\mathrm{m}} \neq \emptyset$ is either the union of straight lines if $d = 2$ or a polygonal surface if $d = 3$, satisfying $\overline{\Sigma} \subsetneq \partial\Omega_{\mathrm{f}}$ and $\overline{\Sigma} \subsetneq \partial\Omega_{\mathrm{m}}$. Then, we let $\Gamma_{\mathrm{f}} := \partial\Omega_{\mathrm{f}} \setminus \overline{\Sigma}$, $\Gamma_{\mathrm{m}} := \partial\Omega_{\mathrm{m}} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_{\mathrm{f}} \cup \Omega_{\mathrm{m}} \cup \Sigma$ and Ω_{f} (and hence inward to Ω_{m} when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t} = \mathbf{t}_1$ when $d = 2$ and by $\mathbf{t}_1, \mathbf{t}_2$, when $d = 3$.

In the free fluid domain Ω_{f}, the motion of the fluid can be described by the following Navier–Stokes/Heat system:

$$\boldsymbol{\sigma}_{\mathrm{f}} = 2\mu \mathbf{e}(\mathbf{u}_{\mathrm{f}}) - p_{\mathrm{f}}\mathbf{I} \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{5.1a}$$

$$-\mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{f}} + (\mathbf{u}_{\mathrm{f}} \cdot \nabla)\mathbf{u}_{\mathrm{f}} - \mathbf{g}_{\mathrm{f}}\theta_{\mathrm{f}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{5.1b}$$

$$\operatorname{div} \mathbf{u}_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{5.1c}$$

$$-\kappa_{\mathrm{f}}\Delta\theta_{\mathrm{f}} + \mathbf{u}_{\mathrm{f}} \cdot \nabla\theta_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}}, \tag{5.1d}$$

where $\mu > 0$ is the dynamic viscosity of the fluid, $\mathbf{u}_{\mathrm{f}}$ is the fluid velocity, p_{f} is the fluid pressure, $\boldsymbol{\sigma}_{\mathrm{f}}$ is the Cauchy stress tensor, $\mathbf{I}$ is the $d \times d$ identity matrix, θ_{f} is the fluid temperature, $\kappa_{\mathrm{f}} > 0$ is the fluid thermal conductivity, $\mathbf{g}_{\mathrm{f}} \in \mathbf{L}^2(\Omega_{\mathrm{f}})$ is the external force per unit mass, $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor, and $\mathbf{e}(\mathbf{u}_{\mathrm{f}})$ is the strain rate tensor given by $\mathbf{e}(\mathbf{u}_{\mathrm{f}}) := \frac{1}{2}\left(\nabla\mathbf{u}_{\mathrm{f}} + (\nabla\mathbf{u}_{\mathrm{f}})^t\right)$, where the superscript t denotes transposition.

In the porous membrane Ω_{m} the behavior of the fluid can be described by the following Darcy-Heat system,

$$\mathbf{K}^{-1}\mathbf{u}_{\mathrm{m}} + \nabla p_{\mathrm{m}} - \mathbf{g}_{\mathrm{m}}\theta_{\mathrm{m}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{5.2a}$$

$$\operatorname{div} \mathbf{u}_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{5.2b}$$

$$-\kappa_{\mathrm{m}}\Delta\theta_{\mathrm{m}} + \mathbf{u}_{\mathrm{m}} \cdot \nabla\theta_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}, \tag{5.2c}$$

where $\mathbf{u}_{\mathrm{m}}$ represents the fluid velocity, p_{m} the fluid pressure, θ_{m} the fluid temperature, $\mathbf{g}_{\mathrm{m}} \in \mathbf{L}^3(\Omega_{\mathrm{m}})$ a given external force, $\kappa_{\mathrm{m}} > 0$ the thermal conductivity, and $\mathbf{K} \in [\mathrm{L}^\infty(\Omega_{\mathrm{m}})]^{d\times d}$ is a symmetric and uniformly positive definite tensor in Ω_{m} representing the intrinsic permeability $\boldsymbol{\kappa}$ of the membrane divided by the dynamic viscosity μ of the fluid.

The transmission conditions that couple the systems (5.1) and (5.2) on the interface Σ are given by

$$\begin{aligned}
&\theta_{\mathrm{f}} = \theta_{\mathrm{m}} && \text{on} \quad \Sigma,\\
&\kappa_{\mathrm{f}}\nabla\theta_{\mathrm{f}} \cdot \mathbf{n} = \kappa_{\mathrm{m}}\nabla\theta_{\mathrm{m}} \cdot \mathbf{n} && \text{on} \quad \Sigma,\\
&\mathbf{u}_{\mathrm{f}} \cdot \mathbf{n} = \mathbf{u}_{\mathrm{m}} \cdot \mathbf{n} && \text{on} \quad \Sigma,\\
&\boldsymbol{\sigma}_{\mathrm{f}}\mathbf{n} + \sum_{j=1}^{d-1} \omega_j(\mathbf{u}_{\mathrm{f}} \cdot \mathbf{t}_j)\mathbf{t}_j = -p_{\mathrm{m}}\mathbf{n} && \text{on} \quad \Sigma,
\end{aligned} \tag{5.3}$$

where $\omega_1, \ldots, \omega_{d-1}$ are positive constants depending on the intrinsic permeability $\boldsymbol{\kappa}$, the viscosity μ, and on the geometrical characteristics of the membrane. In particular, the fourth condition in (5.3) can be decomposed, at least formally, into its normal and tangential components as follows:

$$(\boldsymbol{\sigma}_{\rm f}\mathbf{n})\cdot\mathbf{n} = -p_{\rm m} \quad \text{and} \quad (\boldsymbol{\sigma}_{\rm f}\,\mathbf{n})\cdot\mathbf{t}_j \;=\; -\omega_j(\mathbf{u}_{\rm f}\cdot\mathbf{t}_j) \quad \text{on} \quad \Sigma, \quad j\in 1,\ldots,d-1. \tag{5.4}$$

The first equation in (5.4) corresponds to the balance of normal forces, whereas the second one is known as the Beavers–Joseph–Saffman law, which establishes that the slip velocity along Σ is proportional to the shear stress along Σ (assuming also, based on experimental evidence, that $\mathbf{u}_{\rm m}\cdot\mathbf{t}$ is negligible).

Finally, the Navier–Stokes/Darcy/heat system (5.1), (5.2) and (5.3) is complemented with suitable boundary conditions:

$$\begin{aligned} &\mathbf{u}_{\rm f} = \mathbf{0} \quad &&\text{on} \quad \Gamma_{\rm f}, \qquad &&\mathbf{u}_{\rm m}\cdot\mathbf{n} = 0 \quad &&\text{on} \quad \Gamma_{\rm m},\\ &\theta_{\rm f} = \theta_{\rm D}|_{\Gamma_{\rm f}} \quad &&\text{on} \quad \Gamma_{\rm f}, \qquad &&\theta_{\rm m} = \theta_{\rm D}|_{\Gamma_{\rm m}} \quad &&\text{on} \quad \Gamma_{\rm m}, \end{aligned} \tag{5.5}$$

where $\theta_{\rm D}\in \mathrm{W}^{3/4,4}(\Gamma)$ is a given function defined on $\Gamma := \Gamma_{\rm f}\cup\Gamma_{\rm m}$.

5.2.2 The variational formulation

To derive the variational formulation for the coupled problem given by (5.1), (5.2), (5.3) and (5.5), first we let

$$\begin{aligned} &\mathbf{H}^1_{\Gamma_{\rm f}}(\Omega_{\rm f}) := \Big\{\mathbf{v}\in\mathbf{H}^1(\Omega_{\rm f}) : \quad \mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma_{\rm f}\Big\},\\ &\Psi_\infty := \{\psi\in \mathrm{H}^1(\Omega) \,:\, \psi|_{\Omega_{\rm m}}\in \mathrm{L}^\infty(\Omega_{\rm m})\}, \end{aligned}$$

and define the spaces

$$\mathbf{H} := \{\mathbf{v}\in\mathbf{H}_0(\mathrm{div};\Omega) \,:\, \mathbf{v}|_{\Omega_{\rm f}}\in\mathbf{H}^1_{\Gamma_{\rm f}}(\Omega_{\rm f})\} \quad \text{and} \quad \Psi_{\infty,0} := \Psi_\infty\cap \mathrm{H}^1_0(\Omega),$$

where $\mathbf{H}$ is endowed with the norm

$$\|\mathbf{v}\|_{\mathbf{H}} := \|\mathbf{v}|_{\Omega_{\rm f}}\|_{1,\Omega_{\rm f}} \,+\, \|\mathbf{v}|_{\Omega_{\rm m}}\|_{\mathrm{div};\Omega_{\rm m}} \qquad \forall\,\mathbf{v}\in\mathbf{H}.$$

To derive our weak formulation, first we let $\mathbf{v}\in\mathbf{H}$, multiply (5.1b) and (5.2a) by $\mathbf{v}|_{\Omega_{\rm f}}$ and $\mathbf{v}|_{\Omega_{\rm m}}$, respectively, integrate by parts both equations, employ the identities (5.1a) and (5.4), sum the resulting equations and use the fact that $\mathbf{v}|_{\Omega_{\rm f}}\cdot\mathbf{n} - \mathbf{v}|_{\Omega_{\rm m}}\cdot\mathbf{n} = 0$ on Σ, to obtain

$$\begin{aligned} 2\mu\,(\mathbf{e}(\mathbf{u}_{\rm f}),\mathbf{e}(\mathbf{v}))_{\Omega_{\rm f}} + \sum_{j=1}^{d-1}\omega_j\,\langle\mathbf{u}_{\rm f}\cdot\mathbf{t}_j,\mathbf{v}|_{\Omega_{\rm f}}\cdot\mathbf{t}_j\rangle_\Sigma + ((\mathbf{u}_{\rm f}\cdot\nabla)\,\mathbf{u}_{\rm f},\mathbf{v})_{\Omega_{\rm f}} + (\mathbf{K}_{\rm m}^{-1}\mathbf{u}_{\rm m},\mathbf{v})_{\Omega_{\rm m}}\\ -(p,\mathrm{div}\,\mathbf{v})_\Omega - (\theta_{\rm f}\,\mathbf{g}_{\rm f},\mathbf{v})_{\Omega_{\rm f}} - (\theta_{\rm m}\,\mathbf{g}_{\rm m},\mathbf{v})_{\Omega_{\rm m}} = 0, \end{aligned}$$

for all $\mathbf{v} \in \mathbf{H}$, where $p \in \mathrm{L}^2(\Omega)$ represents the global pressure defined by

$$p := p_\mathrm{f}\chi_\mathrm{f} + p_\mathrm{m}\chi_\mathrm{m},$$

with $\chi_\star$ being the characteristic function:

$$\chi_\star := \begin{cases} 1 & \text{in} \quad \Omega_\star, \\ 0 & \text{in} \quad \Omega \setminus \overline{\Omega}_\star, \end{cases}$$

for $\star \in \{\mathrm{f}, \mathrm{m}\}$. In turn, we let $\psi \in \Psi_{\infty,0}$, multiply (5.1d) and (5.2c) by $\psi|_{\Omega_\mathrm{f}}$ and $\psi|_{\Omega_\mathrm{m}}$, respectively, integrate by parts, use the first and second interface conditions, and sum the resulting equations, to obtain

$$\kappa_\mathrm{f}\,(\nabla\theta_\mathrm{f}, \nabla\psi)_{\Omega_\mathrm{f}} + \kappa_\mathrm{m}\,(\nabla\theta_\mathrm{m}, \nabla\psi)_{\Omega_\mathrm{m}} + (\mathbf{u}_\mathrm{f}\cdot\nabla\theta_\mathrm{f}, \psi)_{\Omega_\mathrm{f}} + (\mathbf{u}_\mathrm{m}\cdot\nabla\theta_\mathrm{m}, \psi)_{\Omega_\mathrm{m}} = 0,$$

for all $\psi \in \Psi_\infty$.

Finally, we incorporate the equations (5.1c) and (5.2b), weakly as follows

$$(q_\mathrm{f}, \operatorname{div}\mathbf{u}_\mathrm{f})_{\Omega_\mathrm{f}} = 0 \quad \text{and} \quad (q_\mathrm{m}, \operatorname{div}\mathbf{u}_\mathrm{m})_{\Omega_\mathrm{m}} = 0,$$

for all $q_\mathrm{f} \in \mathrm{L}^2(\Omega_\mathrm{f})$, $q_\mathrm{m} \in \mathrm{L}^2(\Omega_\mathrm{m})$, respectively.

Then we define the global unknowns

$$\mathbf{u} := \mathbf{u}_\mathrm{f}\chi_\mathrm{f} + \mathbf{u}_\mathrm{m}\chi_\mathrm{m} \quad \text{and} \quad \theta := \theta_\mathrm{f}\chi_\mathrm{f} + \theta_\mathrm{m}\chi_\mathrm{m},$$

and proceed similarly to Chapter 3, to obtain the variational problem:

Find $\mathbf{u} \in \mathbf{H}$, $p \in \mathrm{L}^2_0(\Omega)$ and $\theta \in \mathrm{H}^1(\Omega)$, with $\theta|_\Gamma = \theta_\mathrm{D}$, such that:

$$\begin{aligned} A_\mathrm{F}(\mathbf{u},\mathbf{v}) + O_\mathrm{F}(\mathbf{u};\mathbf{u},\mathbf{v}) + B(\mathbf{v},p) - D(\theta,\mathbf{v}) &= 0 \qquad \forall\,\mathbf{v} \in \mathbf{H}, \\ B(\mathbf{u},q) &= 0 \qquad \forall\,q \in \mathrm{L}^2_0(\Omega), \\ A_\mathrm{T}(\theta,\psi) + O_\mathrm{T}(\mathbf{u};\theta,\psi) &= 0 \qquad \forall\,\psi \in \Psi_{\infty,0}, \end{aligned} \tag{5.6}$$

where the forms $A_\mathrm{F} : \mathbf{H}\times\mathbf{H} \to \mathbb{R}$, $O_\mathrm{F} : \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}) \times \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}) \times \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}) \to \mathbb{R}$, $B : \mathbf{H}\times\mathrm{L}^2_0(\Omega) \to \mathbb{R}$, $D : \mathrm{H}^1(\Omega)\times\mathbf{H} \to \mathbb{R}$, $A_\mathrm{T} : \mathrm{H}^1(\Omega)\times\Psi_{\infty,0} \to \mathbb{R}$, and $O_\mathrm{T} : \mathbf{H}\times\mathrm{H}^1(\Omega)\times\Psi_{\infty,0} \to \mathbb{R}$, are defined respectively, as

$$\begin{aligned} A_\mathrm{F}(\mathbf{u},\mathbf{v}) &:= a_{\mathrm{F},\mathrm{f}}(\mathbf{u},\mathbf{v}) + a_{\mathrm{F},\mathrm{m}}(\mathbf{u},\mathbf{v}), \\ A_\mathrm{T}(\theta,\psi) &:= \kappa_\mathrm{f}(\nabla\theta,\nabla\psi)_{\Omega_\mathrm{f}} + \kappa_\mathrm{m}(\nabla\theta,\nabla\psi)_{\Omega_\mathrm{m}}, \\ B(\mathbf{v},q) &:= -\,(q, \operatorname{div}\mathbf{v})_\Omega, \\ D(\theta,\mathbf{v}) &:= (\theta\,\mathbf{g}_\mathrm{f}, \mathbf{v})_{\Omega_\mathrm{f}} + (\theta\,\mathbf{g}_\mathrm{m}, \mathbf{v})_{\Omega_\mathrm{m}}, \\ O_\mathrm{F}(\mathbf{w};\mathbf{u},\mathbf{v}) &:= ((\mathbf{w}\cdot\nabla)\mathbf{u}, \mathbf{v})_{\Omega_\mathrm{f}}, \\ O_\mathrm{T}(\mathbf{w};\theta,\psi) &:= (\mathbf{w}\cdot\nabla\theta, \psi)_{\Omega_\mathrm{f}} + (\mathbf{w}\cdot\nabla\theta, \psi)_{\Omega_\mathrm{m}}, \end{aligned} \tag{5.7}$$

with

$$a_{\mathrm{F,f}}(\mathbf{u},\mathbf{v}) \;:=\; 2\mu\,(\mathbf{e}(\mathbf{u}),\mathbf{e}(\mathbf{v}))_{\Omega_\mathrm{f}} \;+\; \sum_{j=1}^{d-1}\omega_j\,\langle \mathbf{u}|_{\Omega_\mathrm{f}}\cdot\mathbf{t}_j,\mathbf{v}|_{\Omega_\mathrm{f}}\cdot\mathbf{t}_j\rangle_\Sigma\,,$$
$$a_{\mathrm{F,m}}(\mathbf{u},\mathbf{v}) \;:=\; (\mathbf{K}^{-1}\,\mathbf{u},\mathbf{v})_{\Omega_\mathrm{m}}.$$

We observe that the choice for the pressure p, namely $\mathrm{L}^2_0(\Omega)$, has been made to ensure its uniqueness.

5.2.3 Well-posedness of the continuous problem

To establish the existence, uniqueness and stability result for (5.6), we need to introduce some further notations and results. We will begin by introducing a suitable extension operator, proceeding analogously to Section 3.2.2. To do this, let $\mathrm{E} : \mathrm{W}^{3/4,4}(\Gamma) \to \mathrm{W}^{1,4}(\Omega)$ be the usual lifting operator (see for instance [47, Corollary B.53]), satisfying

$$\gamma_0(\mathrm{E}(\zeta)) = \zeta \quad\text{and}\quad \|\mathrm{E}(\zeta)\|_{1,4,\Omega} \le c\|\zeta\|_{3/4,4,\Gamma} \quad \forall\,\zeta\in\mathrm{W}^{3/4,4}(\Gamma),$$

where $\gamma_0 : \mathrm{W}^{1,4}(\Omega)\to\mathrm{W}^{3/4,4}(\Gamma)$ is the trace operator. In turn, we let $\delta>0$, and similarly to [9, Lemma 2.8], define the function $\beta_\delta : \mathbb{R}^d \to \mathbb{R}$ given by

$$\beta_\delta(\mathbf{x}) := \begin{cases} 1 & \text{if } 0\le \mathrm{dist}(\mathbf{x},\Gamma)\le\delta,\\ 2-\delta^{-1}\mathrm{dist}(\mathbf{x},\Gamma) & \text{if } \delta\le \mathrm{dist}(\mathbf{x},\Gamma)\le 2\delta,\\ 0 & \text{if } \mathrm{dist}(\mathbf{x},\Gamma)\ge 2\delta,\end{cases}$$

where $\mathrm{dist}(\mathbf{x},\Gamma)$ denotes the distance from the point $\mathbf{x}$ to the boundary Γ. Observe that β_δ is continuous and satisfies

$$\beta_\delta\in\mathrm{W}^{1,\infty}(\Omega), \qquad 0\le\beta_\delta\le 1 \quad\text{in}\quad \Omega_\delta,$$
$$\beta_\delta\equiv 0 \quad\text{in}\quad \Omega\setminus\Omega_\delta, \qquad \|\nabla\beta_\delta\|_{0,4,\Omega_\delta}\le\delta^{-1}|\Omega_\delta|^{1/4},$$

where $\Omega_\delta := \{\mathbf{x}\in\Omega : \mathrm{dist}(\mathbf{x},\Gamma)<2\delta\}$, which satisfies $|\Omega_\delta|\le C_\Gamma\delta$, with C_Γ being a positive constant that depends on the measure of Γ. In this way, in order to handle the non-homogeneous Dirichlet boundary condition for the temperature, we introduce the extension operator

$$\mathrm{E}_\delta := \beta_\delta\mathrm{E} : \mathrm{W}^{3/4,4}(\Gamma)\to\mathrm{W}^{1,4}(\Omega), \tag{5.8}$$

which satisfies the following estimates (See Lemma 3.2.3 in Chapter 3):

$$\begin{aligned} \|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} &\le C_{\mathrm{lift},1}\,\delta^{1/12}\,\|\zeta\|_{3/4,4,\Gamma} && \forall\zeta\in\mathrm{W}^{3/4,4}(\Gamma),\\ \|\mathrm{E}_\delta(\zeta)\|_{1,\Omega} &\le C_{\mathrm{lift},2}\,\delta^{-3/4}(1+\delta^2)^{1/2}\,\|\zeta\|_{3/4,4,\Gamma} && \forall\zeta\in\mathrm{W}^{3/4,4}(\Gamma),\\ \|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\mathrm{m}} &\le C_{\mathrm{lift},3}\,(1+\delta^{-2})^{1/2}\|\zeta\|_{3/4,4,\Gamma} && \forall\zeta\in\mathrm{W}^{3/4,4}(\Gamma). \end{aligned} \tag{5.9}$$

Now, we let $\mathbf{V}$ be the kernel of the bilinear form B, that is

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{H} : B(\mathbf{v}, q) = 0 \quad \forall\, q \in \mathrm{L}_0^2(\Omega)\}.$$

From the definition of B we observe that $\mathbf{v} \in \mathbf{V}$ if and only if

$$(q, \operatorname{div} \mathbf{v}|_{\Omega_\mathrm{f}})_{\Omega_\mathrm{f}} + (q, \operatorname{div} \mathbf{v}|_{\Omega_\mathrm{m}})_{\Omega_\mathrm{m}} = 0 \quad \forall\, q \in \mathrm{L}_0^2(\Omega).$$

Then, noting that $\mathbf{v}|_{\Omega_\mathrm{f}} \cdot \mathbf{n} - \mathbf{v}|_{\Omega_\mathrm{m}} \cdot \mathbf{n} = 0$ on Σ and $\mathrm{L}^2(\Omega) = \mathrm{L}_0^2(\Omega) \oplus \mathbb{R}$, it is easy to deduce that

$$\operatorname{div} \mathbf{v}|_{\Omega_\mathrm{f}} = 0 \quad \text{in} \quad \Omega_\mathrm{f} \quad \text{and} \quad \operatorname{div} \mathbf{v}|_{\Omega_\mathrm{m}} = 0 \quad \text{in} \quad \Omega_\mathrm{m}.$$

According to the above, we can rewrite $\mathbf{V}$ as

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{H} : \operatorname{div} \mathbf{v} = 0 \quad \text{in} \quad \Omega\}.$$

Let us now establish the inf-sup condition of the bilinear form B.

Lemma 5.2.1. *There exists $\beta > 0$, such that*

$$\sup_{\mathbf{v} \in \mathbf{H}\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta\, \|q\|_{0,\Omega} \qquad \forall\, q \in \mathrm{L}_0^2(\Omega). \tag{5.10}$$

Proof. The proof follows from the surjectivity of the operator $\operatorname{div} : \mathbf{H}_0^1(\Omega) \to \mathrm{L}_0^2(\Omega)$ and the fact that $\mathbf{H}_0^1(\Omega) \subseteq \mathbf{H}$. We omit further details. □

Next, given a fixed $\delta > 0$, we define the following lifting for the Dirichlet datum $\theta_\mathrm{D} \in \mathrm{W}^{3/4,4}(\Gamma)$:

$$\theta_1 \,:=\, \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega), \tag{5.11}$$

and decompose the unknown $\theta \in \mathrm{H}^1(\Omega)$ as $\theta = \theta_0 + \theta_1$, with $\theta_0 \in \mathrm{H}_0^1(\Omega)$. Then, for the subsequent analysis we introduce the reduced version of problem (5.6) on the kernel $\mathbf{V}$, which reads:

Find $(\mathbf{u}, \theta_0) \in \mathbf{V} \times \mathrm{H}_0^1(\Omega)$, such that

$$\begin{aligned} A_\mathrm{F}(\mathbf{u}, \mathbf{v}) \,+\, O_\mathrm{F}(\mathbf{u}; \mathbf{u}, \mathbf{v}) \,-\, D(\theta_0, \mathbf{v}) \;&=\; D(\theta_1, \mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{V}, \\ A_\mathrm{T}(\theta_0, \psi) \,+\, O_\mathrm{T}(\mathbf{u}; \theta_0 + \theta_1, \psi) \;&=\; -\,A_\mathrm{T}(\theta_1, \psi) && \forall\, \psi \in \Psi_{\infty,0}. \end{aligned} \tag{5.12}$$

It is not difficult to see that problems (5.12) and (5.6) are equivalent. This result is established next.

Lemma 5.2.2. *If* $(\mathbf{u}, p, \theta) \in \mathbf{H} \times \mathrm{L}_0^2(\Omega) \times \mathrm{H}^1(\Omega)$ *is a solution of* (5.6)*, then* $\mathbf{u} \in \mathbf{V}$ *and* $(\mathbf{u}, \theta_0) = (\mathbf{u}, \theta - \theta_1)$*, with* θ_1 *defined in* (5.11) *is a solution to* (5.12)*. Conversely, if* $(\mathbf{u}, \theta_0) \in \mathbf{V} \times \mathrm{H}_0^1(\Omega)$ *is a solution to* (5.12)*, then there exists* $p \in \mathrm{L}_0^2(\Omega)$*, such that* $(\mathbf{u}, p, \theta) = (\mathbf{u}, p, \theta_0 + \theta_1)$ *is a solution to* (5.6)*.*

Proof. The proof follows from the definition of the lifting θ_1 (cf. (5.11)) and the inf-sup condition (5.10). We omit further details. $\square$

At this point, it is important to observe that due to the equivalence between both problems, the analysis of well-posedness for (5.6) is condensed into the examination of problem (5.12). However, since problem (5.12) precisely corresponds to problem (3.26) in Chapter 3, we will now provide a summary of the results obtained in Sections 3.2.2 and 3.2.3 of Chapter 3 to establish the well-posedness of (5.6).

We initiate the discussion by addressing the existence result for problem (5.12). For its proof, we refer the reader to Theorem 3.2.9 in Chapter 3 (see also [20, Theorem 2.9]).

Theorem 5.2.3. *Let* $\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$ *with* $\delta > 0$ *satisfying*

$$c_1\alpha_\mathrm{F}^{-1}\alpha_\mathrm{T}^{-1}\gamma_\mathbf{g}\,\delta^{1/12}\,\|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \le 1,$$

and assume that the lifting θ_1 *and the datum* θ_D *satisfy*

$$c_2\alpha_\mathrm{F}^{-1}\alpha_\mathrm{T}^{-1}\,\gamma_\mathbf{g}\,\|\theta_1\|_{0,\infty,\Omega} \,\le\, 1,$$

and

$$C_\mathbf{u}\,\gamma_\mathbf{g}\,\delta^{-3/4}(1+\delta^2)^{1/2}\,\|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \,\le\, c_3\mu,$$

respectively, where $\gamma_\mathbf{g} := \|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}$, $\alpha_\mathrm{F} := \frac{1}{2}\min\{c_4\mu, C_\mathbf{K}\}$, $\alpha_\mathrm{T} := c_5\min\{\kappa_\mathrm{f},\kappa_\mathrm{m}\}$, $C_\mathbf{u} := c_6\,\alpha_\mathrm{F}^{-1}\alpha_\mathrm{T}^{-1}\,(\max\{\kappa_\mathrm{f},\kappa_\mathrm{m}\}+\alpha_\mathrm{T})$. *Above,* $c_1,\dots,c_6 > 0$ *are constants independent of the physical parameters. Then, there exists at least one solution* $(\mathbf{u},\theta_0) \in \mathbf{V}\times \mathrm{H}_0^1(\Omega)$ *to* (5.12), *which satisfies*

$$\|\mathbf{u}\|_\mathbf{H} \le C_\mathbf{u}\gamma_\mathbf{g}\|\theta_1\|_{1,\Omega} \quad \text{and} \quad \|\theta_0\|_{1,\Omega} \le C_\theta\|\theta_1\|_{1,\Omega}, \tag{5.13}$$

where $C_\theta := \alpha_\mathrm{T}^{-1}\,(2\max\{\kappa_\mathrm{f},\kappa_\mathrm{m}\}+\alpha_\mathrm{T})$.

It is clear that the existence result for (5.6) is a direct consequence of the aforementioned Theorem and Lemma 5.2.2. Moreover, the corresponding estimate for the pressure follows from the inf-sup condition (5.10). We summarize the above in the following corollary and refer to Corollary 3.2.10 in Chapter 3 (see also [20, Corollary 2.10]) for a similar result.

Corollary 5.2.4. *Assume that the hypotheses of Theorem 5.2.3 hold, and let* $(\mathbf{u},\theta_0) \in \mathbf{V}\times\mathrm{H}_0^1(\Omega)$ *be a solution of* (5.12). *There exists* $p \in \mathrm{L}_0^2(\Omega)$, *such that* $(\mathbf{u},p,\theta) = (\mathbf{u},p,\theta_0+\theta_1) \in \mathbf{H}\times\mathrm{L}_0^2(\Omega)\times\mathrm{H}^1(\Omega)$ *is a solution to* (5.6). *In addition, there exist* c_7, $c_8 > 0$, *such that*

$$\|p\|_{0,\Omega} \le (c_7 + c_8\gamma_\mathbf{g}\|\theta_1\|_{1,\Omega})\gamma_\mathbf{g}\|\theta_1\|_{1,\Omega},$$

with $\gamma_\mathbf{g} = \|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}$.

Having this in mind, now we provide the uniqueness of solution of problem (5.6). Its proof can be derived analogously to that of Theorem 3.2.11 in Chapter 3 (see also [20, Theorem 2.11]), hence it is omitted here.

Theorem 5.2.5. *Assume that the hypotheses of Theorem 5.2.3 hold and let* $(\mathbf{u}, p, \theta) = (\mathbf{u}, p, \theta_0 + \theta_1) \in \mathbf{H} \times \mathrm{L}^2_0(\Omega) \times \mathrm{H}^1(\Omega)$ *be a solution of* (5.6). *Assume further that* $\theta_0|_{\Omega_\mathrm{m}} \in \mathrm{W}^{1,4}(\Omega_\mathrm{m})$, *and*

$$\left[(M_1\gamma_\mathbf{g} + M_2)\, C_{\mathrm{lift},2}\delta^{-3/4}(1+\delta^2)^{1/2} + M_3 C_{\mathrm{lift},3}(1+\delta^{-2})^{1/2}\right] \|\theta_\mathrm{D}\|_{3/4,4,\Gamma} + M_3\|\theta_0\|_{1,4,\Omega_\mathrm{m}} + M_4\gamma_\mathbf{g} < 1,$$

where $M_1 := c_9\alpha_\mathrm{F}^{-1} C_\mathbf{u}$, $M_2 := c_{10}\alpha_\mathrm{T}^{-1}\,(C_\theta + 1)$, $M_3 := c_{11}\alpha_\mathrm{T}^{-1}$ *and* $M_4 := c_{12}\alpha_\mathrm{F}^{-1}$, *with* $\gamma_\mathbf{g}$, α_F, $C_\mathbf{u}$, α_T, *and* C_θ *being the parameters defined in Theorem 5.2.3 and* $c_9, \ldots, c_{12} > 0$ *being constants independent of the physical parameters. Then* $(\mathbf{u}, p, \theta)$ *is unique.*

5.3 A mass conservative numerical scheme

In this section we combine an $\mathbf{H}(\mathrm{div})$-conforming scheme for the fluid variables and a conforming Galerkin discretization for the temperature to obtain a mass conservative and pressure robust numerical method to approximate the solution of problem (5.6).

5.3.1 Preliminaries

Let $\mathcal{T}_h^\mathrm{f}$ and $\mathcal{T}_h^\mathrm{m}$ be the respective regular triangulations of the domains Ω_f and Ω_m formed by shape-regular triangles T in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$ of diameter h_T and denote by h_f and h_m their corresponding mesh sizes. Assume that they match in Σ so that $\mathcal{T}_h := \mathcal{T}_h^\mathrm{f} \cup \mathcal{T}_h^\mathrm{m}$ is a triangulation of $\Omega = \Omega_\mathrm{f} \cup \Sigma \cup \Omega_\mathrm{m}$. In addition, we let $h = \max\{h_T : T \in \mathcal{T}_h\}$.

For each $T \in \mathcal{T}_h$, we denote by $\mathbf{n}_T$ the unit outward normal vector on the boundary ∂T and let $\mathcal{E}(T)$ be the set of edges/faces of T. We also denote by $\mathcal{E}_h$ the set of all edges/faces of $\mathcal{T}_h$, subdivided as follows:

$$\mathcal{E}_h := \mathcal{E}_h(\Gamma_\mathrm{f}) \cup \mathcal{E}_h(\Gamma_\mathrm{m}) \cup \mathcal{E}_h(\Omega_\mathrm{f}) \cup \mathcal{E}_h(\Omega_\mathrm{m}) \cup \mathcal{E}_h(\Sigma)\,,$$

where $\mathcal{E}_h(\Gamma_\star) := \{\, F \in \mathcal{E}_h \,:\, F \subseteq \Gamma_\star \,\}$, $\mathcal{E}_h(\Omega_\star) := \{\, F \in \mathcal{E}_h \,:\, F \subseteq \Omega_\star \,\}$ for each $\star \in \{\mathrm{f}, \mathrm{m}\}$, and $\mathcal{E}_h(\Sigma) := \{\, F \in \mathcal{E}_h \,:\, F \subseteq \Sigma \,\}$. In what follows, h_F stands for the diameter of a given edge/face $F \in \mathcal{E}_h$.

We will use standard average and jump operators on $\overline{\Omega}_\mathrm{f}$. To define them, let T^+ and T^- be two adjacent elements of $\mathcal{T}_h^\mathrm{f}$, and $F = \partial T^+ \cap \partial T^- \in \mathcal{E}_h(\Omega_\mathrm{f})$. Let

$\mathbf{v}$ be a piecewise smooth vector–valued function and let us denote by $\mathbf{v}^{\pm}$ its trace taken from within the interior of $T^{\pm}$. Then the jump $[\![\cdot]\!]$ acting on $\mathbf{v}$ is defined as

$$[\![\mathbf{v}]\!] := \begin{cases} \mathbf{v}^{+} \otimes \mathbf{n}_{T^{+}} + \mathbf{v}^{-} \otimes \mathbf{n}_{T^{-}}, & F \in \mathcal{E}_h(\Omega_{\mathrm{f}}), \\ \mathbf{v} \otimes \mathbf{n}, & F \in \mathcal{E}_h(\Gamma_{\mathrm{f}}) \cup \mathcal{E}_h(\Sigma), \end{cases}$$

where $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega_{\mathrm{f}} = \Gamma_{\mathrm{f}} \cup \Sigma$ and $\mathbf{u} \otimes \mathbf{n}$ is the tensor product matrix $\mathbf{u} \otimes \mathbf{n} = (u_i n_j)_{i,j}$. In turn, for any smooth enough piecewise (vector- or tensor-valued) function η, and denoting by $\eta^{\pm}$ its trace taken from within the interior of $T^{\pm}$, we define its average as

$$\{\!\{\eta\}\!\} := \begin{cases} \frac{1}{2}\left(\eta^{+} + \eta^{-}\right), & F \in \mathcal{E}_h(\Omega_{\mathrm{f}}), \\ \eta, & F \in \mathcal{E}_h(\Gamma_{\mathrm{f}}) \cup \mathcal{E}_h(\Sigma). \end{cases}$$

5.3.2 Galerkin Scheme

For $k \geq 1$, let $\mathrm{P}_k(T)$ be the space of polynomials functions on T of degree less or equal than k and inspired by [70, 83], define the following finite element subspaces

$$\begin{aligned} \mathbf{H}_h &:= \{\mathbf{v} \in \mathbf{H}_0(\mathrm{div};\Omega) : \mathbf{v}|_T \in [\mathrm{P}_k(T)]^d \quad \forall\, T \in \mathcal{T}_h\}, \\ \mathrm{Q}_h &:= \{q \in \mathrm{L}^2_0(\Omega) : q|_T \in \mathrm{P}_{k-1}(T) \quad \forall\, T \in \mathcal{T}_h\}, \\ \Psi_h &:= \{\psi \in \mathrm{C}^0(\overline{\Omega}) : \psi|_T \in \mathrm{P}_k(T) \quad \forall\, T \in \mathcal{T}_h\}, \\ \Psi_{h,0} &:= \Psi_h \cap \mathrm{H}^1_0(\Omega). \end{aligned} \tag{5.14}$$

Notice that $\mathbf{H}_h$ corresponds to the well-known Brezzi–Douglas–Marini finite element space [17]. In turn, denoting by $\mathbf{H}_h(\Omega_\star)$ the restriction of $\mathbf{H}_h$ to $\Omega_\star$ for $\star \in \{\mathrm{f}, \mathrm{m}\}$, we observe that $\mathbf{H}_h(\Omega_{\mathrm{m}}) \subseteq \mathbf{H}(\mathrm{div};\Omega_{\mathrm{m}})$, whereas $\mathbf{H}_h(\Omega_{\mathrm{f}})$ is not a subspace of $\mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$. Then, to overcome the nonconformity in Ω_{f}, we proceed similarly to [34] and introduce the following discontinuous versions of the forms $a_{\mathrm{F,f}}$ and O_{F}:

$$\begin{aligned} a^h_{\mathrm{F,f}}(\mathbf{u},\mathbf{v}) :=\ & 2\mu \sum_{T \in \mathcal{T}_h^{\mathrm{f}}} (\mathbf{e}(\mathbf{u}), \mathbf{e}(\mathbf{v}))_T - 2\mu \sum_{F \in \mathcal{E}_h(\Omega_{\mathrm{f}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{f}})} (\{\!\{\mathbf{e}(\mathbf{u})\}\!\}, [\![\mathbf{v}]\!])_F \\ & - 2\mu \sum_{F \in \mathcal{E}_h(\Omega_{\mathrm{f}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{f}})} (\{\!\{\mathbf{e}(\mathbf{v})\}\!\}, [\![\mathbf{u}]\!])_F + 2\mu \sum_{F \in \mathcal{E}_h(\Omega_{\mathrm{f}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{f}})} \frac{a^{\mathrm{pen}}}{h_F} ([\![\mathbf{u}]\!], [\![\mathbf{v}]\!])_F \\ & + \sum_{j=1}^{d-1} \omega_j \sum_{F \in \mathcal{E}_h(\Sigma)} \langle \mathbf{u}|_{\Omega_{\mathrm{f}}} \cdot \mathbf{t}_j, \mathbf{v}|_{\Omega_{\mathrm{f}}} \cdot \mathbf{t}_j \rangle_F , \end{aligned}$$

$$O^h_{\mathrm{F}}(\mathbf{w};\mathbf{u},\mathbf{v}) := \sum_{T \in \mathcal{T}_h^{\mathrm{f}}} ((\mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{v})_T + \sum_{F \in \mathcal{E}_h(\Omega_{\mathrm{f}})} \frac{1}{2} \left(\mathbf{w} \cdot \mathbf{n}_T - |\mathbf{w} \cdot \mathbf{n}_T|, (\mathbf{u}^{ext} - \mathbf{u}) \cdot \mathbf{v}\right)_F .$$

Above, $a^{\mathrm{pen}} > 0$ is the well-known interior penalty parameter chosen on each edge to enforce stability (see, e.g., [2, 42]) whereas $\mathbf{u}^{ext}$ is the trace of $\mathbf{u}$ taken from within the exterior of T. $a^h_{\mathrm{F,f}}$ is the well-known Symmetric Interior Penalty Galerkin discrete form (SIPG) (see [2, 63]) and O^h_{F} is the upwind form introduced by Lasaint–Raviart in [75]. Other choices for $a_{\mathrm{F,f}}$ and O_{F} are equally feasible (see, e.g., [3] and [42, Section 6]), provided that the stability properties in Section 5.3.3 below hold.

To introduce an approximation for the boundary datum θ_{D}, we proceed as in Section 3.3.1, that is, we let $\mathrm{I}_h : \mathrm{C}^0(\overline{\Omega}) \to \Psi_h$ be the well-known Lagrange interpolation operator and recall that, under the assumption $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ and for a given $\delta > 0$, $\mathrm{E}_\delta(\theta_{\mathrm{D}})$ belongs to $\mathrm{W}^{1,4}(\Omega) \subseteq \mathrm{C}^0(\overline{\Omega})$ (cf. (1.6)). Then, for a fixed $\delta > 0$ (to be specified below), we define the following approximation to θ_{D}:

$$\theta^\delta_{\mathrm{D},h} = \mathrm{I}_h(\mathrm{E}_\delta(\theta_{\mathrm{D}}))|_\Gamma \in \{\psi_{\mathrm{D},h} \in \mathrm{C}^0(\overline{\Gamma}) : \psi_{\mathrm{D},h}|_F \in \mathrm{P}_k(F) \quad \forall\, F \in \mathcal{E}_h(\Gamma_{\mathrm{f}}) \cup \mathcal{E}_h(\Gamma_{\mathrm{m}})\}. \tag{5.15}$$

Let us observe that since Ω is a polyhedral domain, Ω_δ is also a polyhedron that can be discretized by shaped-regular elements. According to this, for the forthcoming analysis we let $\mathcal{T}^\delta_h$ be a triangulation of Ω_δ and assume that $\mathcal{T}^\delta_h \subseteq \mathcal{T}_h$.

In this way, we propose the following numerical scheme to approximate the solution of problem (5.6):

Find $(\mathbf{u}_h, p_h, \theta_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$, such that $\theta_h|_\Gamma = \theta^\delta_{\mathrm{D},h}$, and

$$\begin{aligned} A^h_{\mathrm{F}}(\mathbf{u}_h, \mathbf{v}) + O^h_{\mathrm{F}}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) + B(\mathbf{v}, p_h) - D(\theta_h, \mathbf{v}) &= 0 \quad \forall\, \mathbf{v} \in \mathbf{H}_h, \\ B(\mathbf{u}_h, q) &= 0 \quad \forall\, q \in \mathrm{Q}_h, \\ A_{\mathrm{T}}(\theta_h, \psi) + O_{\mathrm{T}}(\mathbf{u}_h; \theta_h, \psi) &= 0 \quad \forall\, \psi \in \Psi_{h,0}, \end{aligned} \tag{5.16}$$

where

$$A^h_{\mathrm{F}}(\mathbf{u}, \mathbf{v}) := a^h_{\mathrm{F,f}}(\mathbf{u}, \mathbf{v}) + a_{\mathrm{F,m}}(\mathbf{u}, \mathbf{v}) \qquad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{H}_h,$$

and $a_{\mathrm{F,m}}$, B, D, A_{T}, and O_{T} are the forms defined in (5.7).

Remark 5.3.1. *If $\mathbf{u}_h \in \mathbf{H}_h$ is the discrete velocity satisfying* (5.16)*, then the second equation of* (5.16) *and the fact that $\mathbf{H}_h \subseteq \mathbf{H}_0(\mathrm{div}; \Omega)$, imply that $\mathrm{div}\, \mathbf{u}_h = 0$ in Ω and $\mathbf{u}_h|_{\Omega_{\mathrm{f}}} \cdot \mathbf{n} = \mathbf{u}_h|_{\Omega_{\mathrm{m}}} \cdot \mathbf{n}$ on Σ, thus the method is strongly mass conservative. In turn, another feasible choice to approximate the velocity is*

$$\mathbf{H}_h := \{\mathbf{v} \in \mathbf{H}_0(\mathrm{div}; \Omega) : \mathbf{v}|_T \in \mathbf{RT}_k(T) \quad \forall\, T \in \mathcal{T}_h\},$$

where $\mathbf{RT}_k(T)$ corresponds to the well-known local Raviart–Thomas space of order k defined by $\mathbf{RT}_k(T) := [\mathrm{P}_k(T)]^d \oplus \mathrm{P}_k(T)\mathbf{x}$, with $\mathbf{x} := (x_1, \ldots, x_d)^t$ is a generic vector of $\mathbb{R}^d$.

5.3.3 Discrete stability properties

Here we discuss the stability properties of the forms involved restricted to the corresponding discrete spaces. To that end, for a given $l \geq 1$, we first define the following broken Sobolev space

$$\mathbf{H}_h^l := \left\{ \mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega) : \mathbf{v}|_T \in \mathbf{H}^l(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{f}} \right\}.$$

On $\mathbf{H}_h^l$, and for each $l = 1, 2$ we define the following norm:

$$\|\mathbf{v}\|_{l,\mathbf{H}_h} := \left(\|\mathbf{v}\|^2_{l,\mathcal{T}_h^{\mathrm{f}}} + \|\mathbf{v}\|^2_{\mathrm{div};\Omega_{\mathrm{m}}} \right)^{1/2} \qquad \forall\, \mathbf{v} \in \mathbf{H}_h^l,$$

where

$$\|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}} := \left(\sum_{T\in\mathcal{T}_h^{\mathrm{f}}} \|\nabla \mathbf{v}\|^2_{0,T} + \sum_{j=1}^{d-1} \sum_{F\in\mathcal{E}_h(\Sigma)} \|\mathbf{v}|_{\Omega_{\mathrm{f}}} \cdot \mathbf{t}_j\|^2_{0,F} + \sum_{F\in\mathcal{E}_h(\Omega_{\mathrm{f}})\cup\mathcal{E}_h(\Gamma_{\mathrm{f}})} \frac{a^{\mathrm{pen}}}{h_F} \|[\![\mathbf{v}]\!]\|^2_{0,F} \right)^{1/2},$$

for all $\mathbf{v} \in \mathbf{H}_h^1$, and

$$\|\mathbf{v}\|_{2,\mathcal{T}_h^{\mathrm{f}}} := \left(\|\mathbf{v}\|^2_{1,\mathcal{T}_h^{\mathrm{f}}} + \sum_{T\in\mathcal{T}_h^{\mathrm{f}}} h_T^2 |\mathbf{v}|^2_{2,T} \right)^{1/2},$$

for all $\mathbf{v} \in \mathbf{H}_h^2$.

In turn, for all $T \in \mathcal{T}_h$, we recall the following inverse and trace inequalities (see, e.g., [42])

$$|\eta|_{2,T} \leq c h_T^{-1} |\eta|_{1,T} \qquad \forall\, \eta \in \mathrm{P}_l(T), \tag{5.17}$$

$$\|\eta\|_{0,\partial T} \leq c(h_T^{-1/2} \|\eta\|_{0,T} + h_T^{1/2} |\eta|_{1,T}) \qquad \forall\, \eta \in \mathrm{H}^1(T), \tag{5.18}$$

$$\|\eta\|_{0,q,\partial T} \leq c h_T^{-1/q} \|\eta\|_{0,q,T} \qquad \forall\, \eta \in \mathrm{P}_l(T), \tag{5.19}$$

where c represents a positive constant independent of the mesh-size. In particular, from (5.17) we obtain that there exists a positive constant c, independent of the mesh-size, such that (see [83, eq. (3.11)])

$$\|\mathbf{v}\|_{2,\mathcal{T}_h^{\mathrm{f}}} \leq c \|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \qquad \forall\, \mathbf{v} \in \mathbf{H}_h. \tag{5.20}$$

This inequality, and estimates (5.18) and (5.19), the latter with $q = 2$, imply (see [3])

$$|A_{\mathrm{F}}^h(\mathbf{u},\mathbf{v})| \leq \widetilde{C}_{A_{\mathrm{F}}} \|\mathbf{u}\|_{2,\mathbf{H}_h} \|\mathbf{v}\|_{1,\mathbf{H}_h} \qquad \forall\, \mathbf{u} \in \mathbf{H}_h^2,\ \forall\, \mathbf{v} \in \mathbf{H}_h, \tag{5.21a}$$

$$|A_{\mathrm{F}}^h(\mathbf{u},\mathbf{v})| \leq \widehat{C}_{A_{\mathrm{F}}} \|\mathbf{u}\|_{1,\mathbf{H}_h} \|\mathbf{v}\|_{1,\mathbf{H}_h} \qquad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{H}_h. \tag{5.21b}$$

In addition, the following estimates are straightforward

$$|B(\mathbf{v},q)| \le \widehat{C}_B \|\mathbf{v}\|_{1,\mathbf{H}_h} \|q\|_{0,\Omega} \qquad \forall\, \mathbf{v} \in \mathbf{H}_h, \forall\, q \in \mathrm{Q}_h, \tag{5.22}$$

$$|A_{\mathrm{T}}(\theta,\psi)| \le \widehat{C}_{A_{\mathrm{T}}} \|\theta\|_{1,\Omega} \|\psi\|_{1,\Omega} \qquad \forall\, \theta,\psi \in \Psi_{h,0}. \tag{5.23}$$

Now, for each $q \ge 1$ if $d = 2$ and $q \in [1,6]$ if $d = 3$ we recall from [71, Proposition 4.5] that the following inequality holds

$$\|\mathbf{v}\|_{0,q,\Omega_{\mathrm{f}}} \le c \|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \qquad \forall\, \mathbf{v} \in \mathbf{H}_h^1, \tag{5.24}$$

where $c > 0$ represents a positive constant independent of h. In addition, from [63, Theorem 4.4] we know that there exists $\widehat{C}_{\mathrm{tr},q} > 0$ independent of h, such that

$$\|\mathbf{v}\|_{0,q,\Sigma} \le \widehat{C}_{\mathrm{tr},q} \|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \qquad \forall\, \mathbf{v} \in \mathbf{H}_h. \tag{5.25}$$

In particular, from (5.24), and after simple computations, we obtain

$$|D(\theta,\mathbf{v})| \le \widehat{C}_D (\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}) \|\theta\|_{1,\Omega} \|\mathbf{v}\|_{1,\mathbf{H}_h} \quad \forall\, \theta \in \Psi_h, \forall\, \mathbf{v} \in \mathbf{H}_h. \tag{5.26}$$

On the other hand, proceeding analogously to the proof of [33, Proposition 4.2] for the two-dimensional case and as in [71, Section 7] for the 3D case (see also [83, Lemma 3.4]), we can obtain the estimate

$$|O_{\mathrm{F}}^h(\mathbf{w}_1;\mathbf{u},\mathbf{v}) - O_{\mathrm{F}}^h(\mathbf{w}_2;\mathbf{u},\mathbf{v})| \le C_{\mathrm{Lip,F}} \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\mathcal{T}_h^{\mathrm{f}}} \|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}}, \tag{5.27}$$

for all $\mathbf{w}_1,\mathbf{w}_2,\mathbf{u} \in \mathbf{H}_h^2$ and for all $\mathbf{v} \in \mathbf{H}_h$, with $C_{\mathrm{Lip,F}} > 0$, being a constant independent of h.

Let us now define the discrete kernel of B as

$$\mathbf{V}_h := \{\mathbf{v} \in \mathbf{H}_h : B(\mathbf{v},q) = 0 \quad \forall\, q \in \mathrm{Q}_h\}.$$

Since the pair $(\mathbf{H}_h, \mathrm{Q}_h)$ satisfies $\mathrm{div}\,(\mathbf{H}_h) \subseteq \mathrm{Q}_h$, it readily follows that (see [34])

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{H}_h : \mathrm{div}\,\mathbf{v} = 0 \quad \text{in} \quad \Omega\}.$$

We observe that, proceeding similarly to Lemma 3.2.2 in Chapter 3, that is, integrating by parts and employing estimate (5.24), we deduce the following result.

Lemma 5.3.1. *The following identity holds true:*

$$O_{\mathrm{T}}(\mathbf{w};\theta,\psi) \;=\; -\,O_{\mathrm{T}}(\mathbf{w};\psi,\theta) \qquad \forall\, \mathbf{w} \in \mathbf{V}_h,\ \forall\, \theta,\psi \in \Psi_{h,0}. \tag{5.28}$$

In addition, for all $\mathbf{w} \in \mathbf{V}_h$ *and for all* θ, $\psi \in \Psi_{h,0}$, *there exist positive constants* $C_{O_{\mathrm{T}}}, \widetilde{C}_{O_{\mathrm{T}}}$, *such that*

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \le C_{O_{\mathrm{T}}} \|\mathbf{w}\|_{1,\mathbf{H}_h} \|\psi\|_{1,\Omega} (\|\theta\|_{0,3,\Omega_{\mathrm{f}}} + \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}), \tag{5.29a}$$

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \le \widetilde{C}_{O_{\mathrm{T}}} \|\mathbf{w}\|_{1,\mathbf{H}_h} \|\psi\|_{1,\Omega} (\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}). \tag{5.29b}$$

Proof. Let $\mathbf{w} \in \mathbf{V}_h$ and $\theta, \psi \in \Psi_{h,0}$ be given. Integrating by parts the two terms defining the form O_{T} (cf. (5.7)) and using the fact that $\operatorname{div} \mathbf{w} = 0$ in Ω and $\mathbf{w}|_{\Omega_{\mathrm{f}}} \cdot \mathbf{n} - \mathbf{w}|_{\Omega_{\mathrm{m}}} \cdot \mathbf{n} = 0$ on Σ, we easily obtain (5.28).

Now, for (5.29a) we employ (5.28) and the Hölder inequality, to obtain

$$\begin{aligned} |O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| &= |O_{\mathrm{T}}(\mathbf{w};\psi,\theta)| \\ &\leq |(\mathbf{w}\cdot\nabla\psi,\theta)_{\Omega_{\mathrm{f}}}| + |(\mathbf{w}\cdot\nabla\psi,\theta)_{\Omega_{\mathrm{m}}}| \\ &\leq \|\mathbf{w}\|_{0,6,\Omega_{\mathrm{f}}} |\psi|_{1,\Omega_{\mathrm{f}}} \|\theta\|_{0,3,\Omega_{\mathrm{f}}} + \|\mathbf{w}\|_{\mathrm{div};\Omega_{\mathrm{m}}} |\psi|_{1,\Omega_{\mathrm{m}}} \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}. \end{aligned}$$

Then, applying (5.24) to $\|\mathbf{w}\|_{0,6,\Omega_{\mathrm{f}}}$ we easily deduce (5.29a). Finally, from (5.29a) and (1.7), the latter applied to $\|\theta\|_{0,3,\Omega_{\mathrm{f}}}$, we easily obtain (5.29b), which concludes the proof. □

Now, noticing that A_{T} is elliptic on $\Psi_{h,0}$ and that O_{T} satisfies (5.28), we easily conclude that for all $\mathbf{w} \in \mathbf{V}_h$, $A_{\mathrm{T}}(\cdot,\cdot) + O_{\mathrm{T}}(\mathbf{w};\cdot,\cdot)$ is elliptic, that is

$$A_{\mathrm{T}}(\psi,\psi) + O_{\mathrm{T}}(\mathbf{w};\psi,\psi) \geq \widehat{\alpha}_{\mathrm{T}} \|\psi\|_{1,\Omega}^2 \qquad \forall\, \psi \in \Psi_{h,0}, \tag{5.30}$$

where $\widehat{\alpha}_{\mathrm{T}}$ is a positive constant independent of h.

Now, we recall from [93, Lemma 2.6] that, for a sufficiently large choice of $a^{\mathrm{pen}} > 0$, there holds

$$a_{\mathrm{F,f}}^h(\mathbf{v},\mathbf{v}) + a_{\mathrm{F,m}}(\mathbf{v},\mathbf{v}) \;\geq\; 2\mu\widehat{\alpha}_{\mathrm{f}} \|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}}^2 + C_{\mathbf{K}} \|\mathbf{v}\|_{\mathrm{div};\Omega_{\mathrm{m}}}^2 \qquad \forall\, \mathbf{v} \in \mathbf{V}_h, \tag{5.31}$$

with $\widehat{\alpha}_{\mathrm{f}} > 0$ independent of h. In addition, we recall from [33] that for any $\mathbf{w} \in \mathbf{V}_h$ and for all $\mathbf{v} \in \mathbf{H}_h$, the form O_{F}^h satisfies

$$O_{\mathrm{F}}^h(\mathbf{w};\mathbf{v},\mathbf{v}) = \frac{1}{2} \sum_{T\in\mathcal{T}_h^{\mathrm{f}}} \big(|\mathbf{w}\cdot\mathbf{n}|, |\mathbf{v}|^2\big)_{0,\partial T\setminus\partial\Omega_{\mathrm{f}}} + \frac{1}{2} \sum_{F\in\mathcal{E}_h(\Sigma)} (\mathbf{w}\cdot\mathbf{n}, |\mathbf{v}|^2)_F. \tag{5.32}$$

Then, combining (5.31) and (5.32) we readily obtain the following result.

Lemma 5.3.2. *Let $\mathbf{w} \in \mathbf{V}_h$, be such that*

$$\|\mathbf{w}\cdot\mathbf{n}\|_{0,\Sigma} \leq 2\mu\widehat{\alpha}_{\mathrm{f}} \widehat{C}_{\mathrm{tr},4}^{-2}. \tag{5.33}$$

Then, there holds

$$A_{\mathrm{F}}^h(\mathbf{v},\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{w};\mathbf{v},\mathbf{v}) \geq \widehat{\alpha}_{\mathrm{F}} \|\mathbf{v}\|_{1,\mathbf{H}_h}^2 \qquad \forall\, \mathbf{v} \in \mathbf{V}_h, \tag{5.34}$$

with $\widehat{\alpha}_{\mathrm{F}} := \frac{1}{2}\min\{\mu\widehat{\alpha}_{\mathrm{f}}, C_{\mathbf{K}}\}$.

Proof. Given $\mathbf{w} \in \mathbf{V}_h$ satisfying (5.33), from (5.31) and (5.32) we obtain

$$A_{\mathrm{F}}^h(\mathbf{v},\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{w};\mathbf{v},\mathbf{v}) \ \geq\ 2\mu\widehat{\alpha}_{\mathrm{f}}\|\mathbf{v}\|_{1,\mathcal{T}_h^{\mathrm{f}}}^2 + C_{\mathbf{K}}\|\mathbf{v}\|_{\mathrm{div};\Omega_{\mathrm{m}}}^2 - \frac{1}{2}\sum_{F\in\mathcal{E}_h(\Sigma)} |(\mathbf{w}\cdot\mathbf{n}, |\mathbf{v}|^2)_F|,$$

for all $\mathbf{v} \in \mathbf{V}_h$. Then, the result follows by applying (5.25) and proceeding analogously to the proof of [45, Lemma 10]. We omit further details. □

Let us now observe that combining [96, Theorem 6.12] and [12, eq. (7.1.28)], it is possible to prove that B satisfies the discrete inf-sup condition

$$\sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} \geq \widehat{\beta}\|q\|_{0,\Omega} \qquad \forall\, q \in \mathrm{Q}_h, \tag{5.35}$$

with $\widehat{\beta} > 0$, independent of h.

Finally, analogously to the continuous case, for a given $\delta > 0$ we introduce the discrete extension operator $\mathrm{E}_{\delta,h} : \mathrm{W}^{3/4,4}(\Gamma) \to \Psi_h$ given by $\mathrm{E}_{\delta,h} := \mathrm{I}_h\mathrm{E}_\delta$, where E_δ is the extension operator defined in (5.8) and I_h is the Lagrange interpolation operator. Then, it is clear from (5.15) that there holds

$$\theta_{\mathrm{D},h}^{\delta} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})|_{\Gamma}. \tag{5.36}$$

Moreover, we recall from Lemma 3.3.4 in Chapter 3, that the aforementioned operator satisfies:

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,3,\Omega} \leq \widehat{C}_{\mathrm{lift},1}\delta^{1/12}(h\delta^{-1} + h + 1)\|\zeta\|_{3/4,4,\Gamma}, \tag{5.37a}$$

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{1,\Omega} \leq \widehat{C}_{\mathrm{lift},2}\delta^{1/4}(2 + \delta^{-1})\|\zeta\|_{3/4,4,\Gamma}, \tag{5.37b}$$

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,\infty,\Omega} \leq (d+1)\|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta}, \tag{5.37c}$$

for all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$, where $\widehat{C}_{\mathrm{lift},1}, \widehat{C}_{\mathrm{lift},2} > 0$ are constants independent of h and δ.

5.3.4 Existence of solution

In this section we establish existence of a discrete solution to problem (5.16) by means of an equivalent fixed-point problem. To that end, we let $\delta > 0$ be fixed (to be specified below in Lemma 5.3.4) and decompose the discrete temperature θ_h as $\theta_h = \theta_{h,0} + \theta_{h,1}$, with $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ and $\theta_{h,0} = \theta_h - \theta_{h,1} \in \Psi_{h,0}$. In turn, to simplify the subsequent analysis, we introduce the reduced version of problem (5.16):

Find $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ such that

$$\begin{array}{rcl} A_{\mathrm{F}}^h(\mathbf{u}_h, \mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) - D(\theta_{h,0}, \mathbf{v}) & = & D(\theta_{h,1}, \mathbf{v}), \\ A_{\mathrm{T}}(\theta_{h,0}, \psi) + O_{\mathrm{T}}(\mathbf{u}_h; \theta_{h,0} + \theta_{h,1}, \psi) & = & -A_{\mathrm{T}}(\theta_{h,1}, \psi), \end{array} \tag{5.38}$$

for all $\mathbf{v} \in \mathbf{V}_h$ and for all $\psi \in \Psi_{h,0}$.

The equivalence between problems (5.16) and (5.38) follows from the discrete inf-sup condition (5.35) and the definition of the lifting $\theta_{h,1}$ (cf. (5.36)). This is established in the following lemma, whose proof is standard and therefore omitted.

Lemma 5.3.3. *If* $(\mathbf{u}_h, p_h, \theta_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ *is a solution of* (5.16), *then* $\mathbf{u}_h \in \mathbf{V}_h$ *and* $(\mathbf{u}_h, \theta_{h,0}) = (\mathbf{u}_h, \theta_h - \theta_{h,1})$ *is a solution of* (5.38). *Conversely, if* $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ *is a solution of* (5.38), *then there exists* $p_h \in \mathrm{Q}_h$*, such that* $(\mathbf{u}_h, p_h, \theta_h) = (\mathbf{u}_h, p_h, \theta_{h,0} + \theta_{h,1})$ *is a solution of* (5.16).

According to the previous lemma, to prove existence of solution of problem (5.6), it suffices to prove solvability of problem (5.38). To do that, now we define the bounded and convex set

$$\mathbf{X}_h := \left\{ \begin{array}{ll} (\mathbf{w}, \phi) \in \mathbf{V}_h \times \Psi_{h,0} : & \|\mathbf{w}\|_{1,\mathbf{H}_h} \leq \widehat{C}_{\mathbf{u}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_{h,1}\|_{1,\Omega} \\ & \text{and } \|\phi\|_{1,\Omega} \leq C_\theta \|\theta_{h,1}\|_{1,\Omega} \end{array} \right\}, \tag{5.39}$$

with

$$\widehat{C}_{\mathbf{u}} := 2\widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{\alpha}_{\mathrm{T}}^{-1}\widehat{C}_D(\widehat{C}_{A_{\mathrm{T}}} + \widehat{\alpha}_{\mathrm{T}}) \quad \text{and} \quad C_\theta := \widehat{\alpha}_{\mathrm{T}}^{-1}(2\widehat{C}_{A_{\mathrm{T}}} + \widehat{\alpha}_{\mathrm{T}}), \tag{5.40}$$

and the discrete operator $\mathcal{J}_h : \mathbf{X}_h \to \mathbf{V}_h \times \Psi_{h,0}$, given by

$$\mathcal{J}_h(\mathbf{w}, \phi) = (\mathbf{u}_h, \theta_{h,0}) \qquad \forall\, (\mathbf{w}, \phi) \in \mathbf{X}_h, \tag{5.41}$$

where $(\mathbf{u}_h, \theta_{h,0})$ is the solution of the linearized version of problem (5.38):

Find $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$, such that

$$\begin{array}{rcl} A_{\mathrm{F}}^h(\mathbf{u}_h, \mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{w}; \mathbf{u}_h, \mathbf{v}) & = & D(\phi, \mathbf{v}) + D(\theta_{h,1}, \mathbf{v}), \\ A_{\mathrm{T}}(\theta_{h,0}, \psi) + O_{\mathrm{T}}(\mathbf{w}; \theta_{h,0}, \psi) & = & -A_{\mathrm{T}}(\theta_{h,1}, \psi) - O_{\mathrm{T}}(\mathbf{w}; \theta_{h,1}, \psi), \end{array} \tag{5.42}$$

for all $\mathbf{v} \in \mathbf{V}_h$ and for all $\psi \in \Psi_{h,0}$.

It is clear that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ is a solution of problem (5.38), if and only if, $\mathcal{J}_h(\mathbf{u}_h, \theta_{h,0}) = (\mathbf{u}_h, \theta_{h,0})$. In this way, to prove solvability of (5.38) in what follows we prove equivalently that $\mathcal{J}_h$ has a fixed-point in $\mathbf{X}_h$ by means of the well-known Brouwer's fixed-point theorem (cf. Theorem 1.6.9).

The following result establishes existence of a fixed-point of $\mathcal{J}_h$.

Lemma 5.3.4. *Let $\delta > 0$ be such that $h \leq \delta$ and*

$$\frac{\widehat{C}_D C_{O_{\mathrm{T}}}}{\widehat{\alpha}_{\mathrm{F}}\widehat{\alpha}_{\mathrm{T}}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\widehat{C}_{\mathrm{lift},1}\delta^{1/12}(\delta+2)\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \leq \frac{1}{4}. \tag{5.43}$$

In addition, assume that $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ and θ_D satisfy, respectively,

$$\frac{\widehat{C}_D C_{O_{\mathrm{T}}}}{\widehat{\alpha}_{\mathrm{F}}\widehat{\alpha}_{\mathrm{T}}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_{h,1}\|_{0,\infty,\Omega} \leq \frac{1}{4}, \tag{5.44}$$

and

$$\widehat{C}_{\mathrm{tr},2}\widehat{C}_{\mathbf{u}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\widehat{C}_{\mathrm{lift},2}\delta^{1/4}(2+\delta^{-1})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \leq 2\mu\widehat{\alpha}_{\mathrm{f}}\widehat{C}_{\mathrm{tr},4}^{-2}. \tag{5.45}$$

Then, $\mathcal{J}_h$ is well defined and there exists at least one $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$, such that $\mathcal{J}_h(\mathbf{u}_h, \theta_{h,0}) = (\mathbf{u}_h, \theta_{h,0})$.

Proof. For a given $(\mathbf{w}, \phi) \in \mathbf{X}_h$, we note that using the trace inequality (5.25) and the estimates (5.37b) and (5.45), we get

$$\|\mathbf{w}\cdot\mathbf{n}\|_{0,\Sigma} \leq \widehat{C}_{\mathrm{tr},2}\|\mathbf{w}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \leq \widehat{C}_{\mathrm{tr},2}\|\mathbf{w}\|_{1,\mathbf{H}_h} \leq \widehat{C}_{\mathrm{tr},2}\widehat{C}_{\mathbf{u}}\gamma_{\mathbf{g}}\|\theta_{h,1}\|_{1,\Omega} \leq 2\mu\widehat{\alpha}_{\mathrm{f}}\widehat{C}_{\mathrm{tr},4}^{-2}, \tag{5.46}$$

where for the sake of simplicity, we use the notation $\gamma_{\mathbf{g}} := \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$. The latter implies that $\mathbf{w}$ satisfies (5.33). Then, from (5.30) and (5.34) and the Lax–Milgram lemma, we conclude the unique solvability of (5.42), which implies the existence of a unique $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$, such that $\mathcal{J}_h(\mathbf{w}, \phi) = (\mathbf{u}_h, \theta_{h,0})$.

Now we turn to prove that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$. To do that, we notice first that assumption $h \leq \delta$ implies that estimate (5.37a) with $\zeta = \theta_{\mathrm{D}}$ becomes

$$\|\theta_{h,1}\|_{0,3,\Omega} = \|\mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})\|_{0,3,\Omega} \leq \widehat{C}_{\mathrm{lift},1}\delta^{1/12}(\delta+2)\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}. \tag{5.47}$$

Then, from (5.30), (5.34), (5.42), (5.43), (5.44), (5.47), and using the definition of $C_{\mathbf{u}}$ and C_θ (cf. (5.40)), it is possible to prove that $(\mathbf{u}_h, \theta_{h,0})$ satisfies

$$\begin{aligned}
\|\mathbf{u}_h\|_{1,\mathbf{H}_h} &\leq \widehat{\alpha}_{\mathrm{F}}^{-1}\,\widehat{C}_D\gamma_{\mathbf{g}}\,(\,\|\phi\|_{1,\Omega} + \|\theta_{h,1}\|_{1,\Omega}\,)\\
&\leq \widehat{\alpha}_{\mathrm{F}}^{-1}\,\widehat{C}_D\gamma_{\mathbf{g}}\,(C_\theta+1)\,\|\theta_{h,1}\|_{1,\Omega}\\
&\leq \widehat{C}_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,\|\theta_{h,1}\|_{1,\Omega},
\end{aligned}$$

and

$$\begin{aligned}
&\|\theta_{h,0}\|_{1,\Omega} \leq \widehat{\alpha}_{\mathrm{T}}^{-1}\left[\widehat{C}_{A_{\mathrm{T}}}\|\theta_{h,1}\|_{1,\Omega} + C_{O_{\mathrm{T}}}\|\mathbf{u}\|_{\mathbf{H}}\left(C_{\mathrm{lift},1}\delta^{1/12}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + \|\theta_1\|_{0,\infty,\Omega}\right)\right]\\
&\leq \widehat{\alpha}_{\mathrm{T}}^{-1}\left[\widehat{C}_{A_{\mathrm{T}}}\|\theta_{h,1}\|_{1,\Omega} + C_{O_{\mathrm{T}}}\widehat{C}_{\mathbf{u}}\gamma_{\mathbf{g}}\|\theta_{h,1}\|_{1,\Omega}\left(C_{\mathrm{lift},1}\delta^{1/12}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + \|\theta_{h,1}\|_{0,\infty,\Omega}\right)\right]\\
&\leq C_\theta\|\theta_{h,1}\|_{1,\Omega},
\end{aligned}$$

which implies that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$, thus $\mathcal{J}_h(\mathbf{X}_h) \subseteq \mathbf{X}_h$. According to the above, it follows that $\mathcal{J}_h$ is well defined.

To prove now that $\mathcal{J}_h$ is a continuous mapping, we let $(\mathbf{w}, \phi) \in \mathbf{X}_h$ and $\{(\mathbf{w}_j, \phi_j)\}_{j\in\mathbb{N}} \subseteq \mathbf{X}_h$, be such that $\|\mathbf{w}_j - \mathbf{w}\|_{1,\mathbf{H}_h} \stackrel{j\to\infty}{\longrightarrow} 0$ and $\|\phi_j - \phi\|_{1,\Omega} \stackrel{j\to\infty}{\longrightarrow} 0$, and let $\{(\mathbf{u}_j, \theta_j)\}_{j\in\mathbb{N}} \subseteq \mathbf{X}_h$ and $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$ given, respectively, by

$$\mathcal{J}_h(\mathbf{w}_j, \phi_j) = (\mathbf{u}_j, \theta_j) \quad \forall\, j \in \mathbb{N} \quad \text{and} \quad \mathcal{J}_h(\mathbf{w}, \phi) = (\mathbf{u}_h, \theta_{h,0}).$$

To prove the continuity of $\mathcal{J}_h$ it suffices to prove that $\|\mathbf{u}_j - \mathbf{u}_h\|_{1,\mathbf{H}_h} \stackrel{j\to\infty}{\longrightarrow} 0$ and $\|\theta_j - \theta_{h,0}\|_{1,\Omega} \stackrel{j\to\infty}{\longrightarrow} 0$. To that end, given $j \in \mathbb{N}$, from (5.42) and the definition of $\mathcal{J}_h$ (cf. (5.41)), we first observe that

$$\begin{aligned} A^h_{\mathrm{F}}(\mathbf{u}_h - \mathbf{u}_j, \mathbf{v}) + O^h_{\mathrm{F}}(\mathbf{w}; \mathbf{u}_h, \mathbf{v}) - O^h_{\mathrm{F}}(\mathbf{w}_j; \mathbf{u}_j, \mathbf{v}) &= D(\phi - \phi_j, \mathbf{v}), \\ A_{\mathrm{T}}(\theta_{h,0} - \theta_j, \psi) + O_{\mathrm{T}}(\mathbf{w}; \theta_{h,0}, \psi) - O_{\mathrm{T}}(\mathbf{w}_j; \theta_j, \psi) &= -O_{\mathrm{T}}(\mathbf{w} - \mathbf{w}_j; \theta_{h,1}, \psi), \end{aligned} \tag{5.48}$$

for all $\mathbf{v} \in \mathbf{V}_h$ and for all $\psi \in \Psi_{h,0}$. In turn, noticing that $\mathbf{w}$ satisfies (5.33) (cf. (5.46)), we have that $A^h_{\mathrm{F}}(\cdot,\cdot) + O^h_{\mathrm{F}}(\mathbf{w};\cdot,\cdot)$ is elliptic (cf. (5.34)) on $\mathbf{V}_h$. Then, from the first equation in (5.48) with $\mathbf{v} = \mathbf{u}_h - \mathbf{u}_j$, adding and subtracting suitable terms, employing the continuity of O^h_{F} and D, and the fact that $\mathbf{w}_j \stackrel{j\to\infty}{\longrightarrow} \mathbf{w}$ and $\phi_j \stackrel{j\to\infty}{\longrightarrow} \phi$, we arrive at

$$\widehat{\alpha}_{\mathrm{F}}\, \|\mathbf{u}_h - \mathbf{u}_j\|_{1,\mathbf{H}_h} \le C_{\mathrm{Lip,F}} \|\mathbf{w} - \mathbf{w}_j\|_{1,\mathbf{H}_h} \|\mathbf{u}_h\|_{1,\mathcal{T}^{\mathrm{f}}_h} + \widehat{C}_D \gamma_{\mathbf{g}} \|\phi - \phi_j\|_{1,\Omega} \stackrel{j\to\infty}{\longrightarrow} 0. \tag{5.49}$$

Similarly, in the second equation of (5.48), we let $\psi = \theta_h - \theta_j$, and after a suitable algebraic manipulations, using (5.29b) and (5.30), and the fact $\mathbf{w}_j \stackrel{j\to\infty}{\longrightarrow} \mathbf{w}$, we obtain

$$\widehat{\alpha}_{\mathrm{T}}\, \|\theta_h - \theta_j\|_{1,\Omega} \le \widetilde{C}_{O_{\mathrm{T}}}\, \|\mathbf{w} - \mathbf{w}_j\|_{1,\mathbf{H}_h} \left(\|\theta_{h,0} + \theta_{h,1}\|_{1,\Omega_{\mathrm{f}}} + \|\theta_{h,0} + \theta_{h,1}\|_{0,\infty,\Omega_{\mathrm{m}}} \right) \stackrel{j\to\infty}{\longrightarrow} 0. \tag{5.50}$$

In this way, according to the definition of $\mathcal{J}_h$ (cf. (5.41)), from (5.49) and (5.50) we obtain that $\mathcal{J}_h(\mathbf{w}_j, \phi_j) \stackrel{j\to\infty}{\longrightarrow} \mathcal{J}_h(\mathbf{w}, \phi)$, which implies the continuity of $\mathcal{J}_h$.

In this way, applying the Brouwer fixed-point Theorem (cf. Theorem 1.6.9) it follows that there exists at least one $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$, such that $(\mathbf{u}_h, \theta_{h,0}) = \mathcal{J}_h(\mathbf{u}_h, \theta_{h,0})$. □

We end this section by establishing the existence of solution of problem (5.16) and its stability.

Theorem 5.3.5. *Assume that the hypotheses of Lemma 5.3.4 hold. Then, there exists at least one* $(\mathbf{u}_h, p_h, \theta_h) = (\mathbf{u}_h, p_h, \theta_{h,0} + \theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ *solution to*

(5.16). *Moreover, there exist positive constants* $\widehat{c}_1$, $\widehat{c}_2$, $\widehat{c}_3$, $\widehat{c}_4$, *independent of the solution and* h, *such that*

$$\begin{aligned} \|\mathbf{u}_h\|_{1,\mathbf{H}_h} &\leq \widehat{c}_1 \gamma_{\mathbf{g}} \delta^{1/4}(2+\delta^{-1})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}, \\ \|\theta_h\|_{1,\Omega} &\leq \widehat{c}_2 \delta^{1/4}(2+\delta^{-1})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}, \end{aligned} \tag{5.51}$$

and

$$\|p_h\|_{0,\Omega} \leq (\widehat{c}_3 + \widehat{c}_4 \gamma_{\mathbf{g}} \delta^{1/4}(2+\delta^{-1})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma})\gamma_{\mathbf{g}}\delta^{1/4}(2+\delta^{-1})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}, \tag{5.52}$$

with $\gamma_{\mathbf{g}} = \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$.

Proof. The existence of a solution to problem (5.16) follows directly from Lemmas 5.3.3 and 5.3.4. To derive estimate (5.51), we utilize the fact that $\mathbf{u}_h$ and $\theta_{h,0}$ belong to $\mathbf{X}_h$ (cf. (5.39)) along with (5.37b) and the fact that $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})$. For p_h, the discrete inf-sup condition (5.35) is applied in combination with the first equation of (5.16) and estimates (5.51). □

5.4 Error analysis

In this section, we carry out the error analysis of the finite element scheme (5.16). To that end, from now on we assume that the hypotheses of Theorem 5.2.5 hold and let $(\mathbf{u}, p, \theta) = (\mathbf{u}, p, \theta_0 + \theta_1) \in \mathbf{H} \times \mathrm{L}^2_0(\Omega) \times \mathrm{H}^1(\Omega)$ be the unique solution of problem (5.6), with $\theta_1 = \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega)$ and $\theta_0 \in \mathrm{H}^1_0(\Omega)$. In addition, we assume that the hypotheses of Theorem 5.3.5 hold and let $(\mathbf{u}_h, p_h, \theta_h) = (\mathbf{u}_h, p_h, \theta_{h,0}+\theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ be a solution of (5.16), with $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ and $\theta_{h,0} \in \Psi_{h,0}$. In addition, given $k \geq 1$, in what follows we assume that the exact solution satisfies:

$$\begin{gathered} \mathbf{u} \in \left\{ \mathbf{v} \in \mathbf{H} \,:\, \mathbf{v}|_{\Omega_{\mathrm{f}}} \in \mathbf{H}^{k+1}(\Omega_{\mathrm{f}}), \quad \mathbf{v}|_{\Omega_{\mathrm{m}}} \in \mathbf{H}^k(\Omega_{\mathrm{m}}) \quad \text{and} \quad \operatorname{div}(\mathbf{v}|_{\Omega_{\mathrm{m}}}) \in \mathrm{H}^k(\Omega_{\mathrm{m}}) \right\}, \\ p \in \mathrm{H}^k(\Omega), \quad \theta \in \mathrm{H}^{k+1}(\Omega), \quad \theta|_{\Omega_{\mathrm{m}}} \in \mathrm{W}^{k+1,4}(\Omega_{\mathrm{m}}). \end{gathered} \tag{5.53}$$

Then, we let $\boldsymbol{\Pi}_h : \mathbf{H}^1(\Omega) \to \mathbf{H}_h$ be the BDM interpolation, $\mathcal{P}_h : \mathrm{L}^2(\Omega) \to \mathrm{Q}_h$ the L^2-projection, and $\mathrm{I}_h : \mathrm{C}(\overline{\Omega}) \to \Psi_h$ the Lagrange interpolator, and write the corresponding errors as

$$\mathbf{e}_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_h, \qquad \mathrm{e}_p = p - p_h \quad \text{and} \quad \mathrm{e}_\theta = \theta - \theta_h.$$

Then, we decompose these errors as follows

$$\begin{aligned} \mathbf{e}_{\mathbf{u}} &= \boldsymbol{\varrho}_{\mathbf{u}} + \boldsymbol{\chi}_{\mathbf{u}}, & \boldsymbol{\varrho}_{\mathbf{u}} &= \mathbf{u} - \boldsymbol{\Pi}_h(\mathbf{u}), & \boldsymbol{\chi}_{\mathbf{u}} &= \boldsymbol{\Pi}_h(\mathbf{u}) - \mathbf{u}_h, \\ \mathrm{e}_p &= \varrho_p + \chi_p, & \varrho_p &= p - \mathcal{P}_h(p), & \chi_p &= \mathcal{P}_h(p) - p_h, \\ \mathrm{e}_\theta &= \varrho_\theta + \chi_\theta, & \varrho_\theta &= \theta - \mathrm{I}_h(\theta), & \chi_\theta &= \mathrm{I}_h(\theta) - \theta_h. \end{aligned} \tag{5.54}$$

With the above definitions, we observe from [12, Chapter 2], [52, Chapter 3], and [47, Section 1.5] that the following estimates hold

$$\begin{aligned}
&\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\mathrm{f}}} \le c h^k \|\mathbf{u}\|_{k+1,\Omega_{\mathrm{f}}},\\
&\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} \le c h^k (\|\mathbf{u}\|_{k,\Omega_{\mathrm{m}}} + \|\operatorname{div}\mathbf{u}\|_{k,\Omega_{\mathrm{m}}}),\\
&\|\varrho_p\|_{0,\Omega} \le c h^k \|p\|_{k,\Omega}, \qquad (5.55)\\
&\|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} \le c h^k \|\theta\|_{k+1,\Omega_{\mathrm{f}}},\\
&\|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}} \le c h^k \|\theta\|_{k+1,4,\Omega_{\mathrm{m}}}.
\end{aligned}$$

Finally, the following estimate for O_{T} will be employed next in the error analysis for a given $\mathbf{w} \in \mathbf{H}_h$, whose proof can be obtained proceeding analogously to the proof of Lemma 3.4.1 in Chapter 3: There exists $\bar{C}_{O_{\mathrm{T}}} > 0$, independent of h, such that

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \le \bar{C}_{O_{\mathrm{T}}} \|\mathbf{w}\|_{1,\mathbf{H}_h} (\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{1,4,\Omega_{\mathrm{m}}}) \|\psi\|_{1,\Omega}, \qquad (5.56)$$

for all $\theta \in \{\psi \in \mathrm{H}^1(\Omega) : \psi|_{\Omega_{\mathrm{m}}} \in \mathrm{W}^{1,4}(\Omega_{\mathrm{m}})\}$ and $\psi \in \Psi_{h,0}$.

Now we are in position of establishing the main result of this section, namely, the theoretical rate of convergence for the Galerkin scheme (5.16). We observe in advance that the estimate for the velocity (cf. (5.58)) does not depend on the pressure, thus confirming that the method is pressure robust.

Theorem 5.4.1. *Assume that the hypotheses in Theorems 5.2.5 and 5.3.5 hold. Let* $(\mathbf{u}, p, \theta) = (\mathbf{u}, p, \theta_0 + \theta_1) \in \mathbf{H} \times \mathrm{L}_0^2(\Omega) \times \mathrm{H}^1(\Omega)$ *be the unique solution to* (5.6), *with* $\theta_1 = \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega)$ *and* $\theta_0 \in \mathrm{H}_0^1(\Omega)$, *and assume that* (5.53) *holds. In addition, let* $(\mathbf{u}_h, p_h, \theta_h) = (\mathbf{u}_h, p_h, \theta_{h,0} + \theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ *be a solution to* (5.16), *with* $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ *and* $\theta_{h,0} \in \Psi_{h,0}$. *Finally, let* $\gamma_{\mathbf{g}} = \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$, *and assume further that*

$$C_1 \delta^{-3/4} (1+\delta^2)^{1/2} \gamma_{\mathbf{g}} \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + C_2 \gamma_{\mathbf{g}} \|\theta\|_{1,4,\Omega_{\mathrm{m}}} \le \frac{1}{2}, \qquad (5.57)$$

where C_1 *and* C_2 *are positive constants (defined in* (5.65)*) independent of* h. *Then, there exist* $C_{rate,1}$, $C_{rate,2} > 0$, *independent of* h *and the continuous and discrete solutions, such that*

$$\|\mathbf{e}_{\mathbf{u}}\|_{1,\mathbf{H}_h} + \|\mathrm{e}_\theta\|_{1,\Omega} \le C_{rate,1} h^k \Big\{ \|\mathbf{u}\|_{k+1,\Omega_{\mathrm{f}}} + \|\mathbf{u}\|_{k,\Omega_{\mathrm{m}}} + \|\theta\|_{k+1,\Omega_{\mathrm{f}}} + \|\theta\|_{k+1,4,\Omega_{\mathrm{m}}} \Big\}, \qquad (5.58)$$

and

$$\|\mathrm{e}_p\|_{0,\Omega} \le C_{rate,2} h^k \Big\{ \|\mathbf{u}\|_{k+1,\Omega_{\mathrm{f}}} + \|\mathbf{u}\|_{k,\Omega_{\mathrm{m}}} + \|p\|_{k,\Omega} + \|\theta\|_{k+1,\Omega_{\mathrm{f}}} + \|\theta\|_{k+1,4,\Omega_{\mathrm{m}}} \Big\}. \qquad (5.59)$$

Proof. We begin by noticing that, owing to the extra regularity of the exact solution, we have

$$A_{\mathrm{F}}^h(\mathbf{u},\mathbf{v}) = A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) \quad \text{and} \quad O_{\mathrm{F}}^h(\mathbf{u};\mathbf{u},\mathbf{v}) = O_{\mathrm{F}}(\mathbf{u};\mathbf{u},\mathbf{v}), \qquad \forall\, \mathbf{v}\in\mathbf{H}_h.$$

Then, the following orthogonality property holds:

$$\begin{aligned} A_{\mathrm{F}}^h(\mathbf{e_u},\mathbf{v}) + \left[O_{\mathrm{F}}^h(\mathbf{u};\mathbf{u},\mathbf{v}) - O_{\mathrm{F}}^h(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v})\right] + B(\mathbf{v},\mathrm{e}_p) - D(\mathrm{e}_\theta,\mathbf{v}) &= 0,\\ B(\mathbf{e_u},q) &= 0, \\ A_{\mathrm{T}}(\mathrm{e}_\theta,\psi) + [O_{\mathrm{T}}(\mathbf{u};\theta,\psi) - O_{\mathrm{T}}(\mathbf{u}_h;\theta_h,\psi)] &= 0, \end{aligned} \tag{5.60}$$

for all $\mathbf{v}\in\mathbf{H}_h$, for all $q\in\mathrm{Q}_h$ and for all $\psi\in\Psi_{h,0}$. In particular, from the first equation of (5.60), adding and subtracting suitable terms and utilizing the decomposition (5.54), we obtain

$$\begin{aligned} A_{\mathrm{F}}^h(\boldsymbol{\chi}_\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{u}_h;\boldsymbol{\chi}_\mathbf{u},\mathbf{v}) = {}& -A_{\mathrm{F}}^h(\boldsymbol{\varrho}_\mathbf{u},\mathbf{v}) - O_{\mathrm{F}}^h(\mathbf{u}_h;\boldsymbol{\varrho}_\mathbf{u},\mathbf{v})\\ & + D(\chi_\theta,\mathbf{v}) + D(\varrho_\theta,\mathbf{v}) - \left[O_{\mathrm{F}}^h(\mathbf{u};\mathbf{u},\mathbf{v}) - O_{\mathrm{F}}^h(\mathbf{u}_h;\mathbf{u},\mathbf{v})\right], \end{aligned}$$

for all $\mathbf{v}\in\mathbf{V}_h$. From this identity with $\mathbf{v}=\boldsymbol{\chi}_\mathbf{u}$, and employing the coercivity of $A_{\mathrm{F}}^h(\cdot,\cdot)+O_{\mathrm{F}}^h(\mathbf{u}_h;\cdot,\cdot)$ (cf. (5.34)), the continuity of A_{F}^h (cf. (5.21a)), the continuity of D (cf. (5.26)), and the Lipschitz continuity of O_{F}^h (cf. (5.27)), we obtain

$$\widehat{\alpha}_{\mathrm{F}}\|\boldsymbol{\chi}_\mathbf{u}\|_{1,\mathbf{H}_h} \le \widehat{C}_D\gamma_\mathbf{g}\|\chi_\theta\|_{1,\Omega} + C_{\mathrm{Lip,F}}\|\boldsymbol{\chi}_\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}}\|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}} + L_1, \tag{5.61}$$

where

$$L_1 := \widetilde{C}_{A_{\mathrm{F}}}\|\boldsymbol{\varrho}_\mathbf{u}\|_{2,\mathbf{H}_h} + C_{\mathrm{Lip,F}}(\|\mathbf{u}_h\|_{1,\mathcal{T}_h^{\mathrm{f}}} + \|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}})\|\boldsymbol{\varrho}_\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}} + \widehat{C}_D\gamma_\mathbf{g}\|\varrho_\theta\|_{1,\Omega}.$$

Similarly, from the third equation of (5.60), we obtain

$$\begin{aligned} A_{\mathrm{T}}(\chi_\theta,\psi) + O_{\mathrm{T}}(\mathbf{u}_h;\chi_\theta,\psi) = {}& -A_{\mathrm{T}}(\varrho_\theta,\psi) - O_{\mathrm{T}}(\mathbf{u}_h;\varrho_\theta,\psi)\\ & - [O_{\mathrm{T}}(\mathbf{u};\theta,\psi) - O_{\mathrm{T}}(\mathbf{u}_h;\theta,\psi)], \end{aligned}$$

for all $\psi\in\Psi_{h,0}$, and in particular taking $\psi=\chi_\theta$, and making use of estimates (5.23), (5.30), (5.56), to obtain

$$\widehat{\alpha}_{\mathrm{T}}\|\chi_\theta\|_{1,\Omega} \le \bar{C}_{O_{\mathrm{T}}}\|\boldsymbol{\chi}_\mathbf{u}\|_{1,\mathbf{H}_h}\left(\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{1,4,\Omega_{\mathrm{m}}}\right) + L_2, \tag{5.62}$$

where

$$\begin{aligned} L_2 := {}& \widehat{C}_{A_{\mathrm{T}}}\|\varrho_\theta\|_{1,\Omega} + \bar{C}_{O_{\mathrm{T}}}\|\mathbf{u}_h\|_{1,\mathbf{H}_h}\left(\|\varrho_\theta\|_{1,\Omega_{\mathrm{f}}} + \|\varrho_\theta\|_{1,4,\Omega_{\mathrm{m}}}\right)\\ & + \bar{C}_{O_{\mathrm{T}}}\|\boldsymbol{\varrho}_\mathbf{u}\|_{1,\mathbf{H}_h}\left(\|\theta\|_{1,\Omega_{\mathrm{f}}} + \|\theta\|_{1,4,\Omega_{\mathrm{m}}}\right). \end{aligned}$$

Then, combining (5.61) and (5.62), it follows that

$$\begin{aligned}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \leq\ & \widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{C}_D\gamma_{\mathbf{g}}\widehat{\alpha}_{\mathrm{T}}^{-1}\bar{C}_{O_{\mathrm{T}}}\|\theta\|_{1,\Omega_{\mathrm{f}}}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \\ & + \widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{C}_D\gamma_{\mathbf{g}}\widehat{\alpha}_{\mathrm{T}}^{-1}\bar{C}_{O_{\mathrm{T}}}\|\theta\|_{1,4,\Omega_{\mathrm{m}}}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \\ & + \widehat{\alpha}_{\mathrm{F}}^{-1}C_{\mathrm{Lip,F}}\|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} + \widehat{\alpha}_{\mathrm{F}}^{-1}L_1 + \widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{C}_D\gamma_{\mathbf{g}}\widehat{\alpha}_{\mathrm{T}}^{-1}L_2.\end{aligned} \tag{5.63}$$

Now, from (5.13) and using the second estimate of the continuous lifting (cf. (5.9)), we observe that $\mathbf{u}$ and θ satisfy

$$\begin{aligned}&\|\mathbf{u}\|_{1,\mathcal{T}_h^{\mathrm{f}}} \leq \|\mathbf{u}\|_{1,\Omega_{\mathrm{f}}} \leq \|\mathbf{u}\|_{\mathbf{H}} \leq C_{\mathbf{u}}C_{\mathrm{lift},2}\delta^{-3/4}(1+\delta^2)^{1/2}\gamma_{\mathbf{g}}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}, \\ &\|\theta\|_{1,\Omega_{\mathrm{f}}} \leq \|\theta\|_{1,\Omega} \leq (C_\theta+1)C_{\mathrm{lift},2}\delta^{-3/4}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}.\end{aligned} \tag{5.64}$$

Hence, from (5.63) and (5.64), we obtain

$$\begin{aligned}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \leq\ & (C_1\delta^{-3/4}(1+\delta^2)^{1/2}\gamma_{\mathbf{g}}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + C_2\gamma_{\mathbf{g}}\|\theta\|_{1,4,\Omega_{\mathrm{m}}})\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \\ & + \widehat{\alpha}_{\mathrm{F}}^{-1}L_1 + \widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{C}_D\gamma_{\mathbf{g}}\widehat{\alpha}_{\mathrm{T}}^{-1}L_2,\end{aligned}$$

with

$$\begin{aligned}C_1 &:= \widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{C}_D\widehat{\alpha}_{\mathrm{T}}^{-1}\bar{C}_{O_{\mathrm{T}}}(C_\theta+1)C_{\mathrm{lift},2} + \widehat{\alpha}_{\mathrm{F}}^{-1}C_{\mathrm{Lip,F}}C_{\mathbf{u}}C_{\mathrm{lift},2}, \\ C_2 &:= \widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{C}_D\widehat{\alpha}_{\mathrm{T}}^{-1}\bar{C}_{O_{\mathrm{T}}}.\end{aligned} \tag{5.65}$$

Which together with assumption (5.57), implies

$$\|\boldsymbol{\chi}_{\mathbf{u}}\|_{1,\mathbf{H}_h} \leq c_1L_1 + c_2L_2, \tag{5.66}$$

with $c_1 := 2\widehat{\alpha}_{\mathrm{F}}^{-1}$ and $c_2 := 2\widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{C}_D\gamma_{\mathbf{g}}\widehat{\alpha}_{\mathrm{T}}^{-1}$. Moreover, replacing (5.66) in (5.62), and employing (5.57) and the estimate for θ in (5.13), we deduce

$$\|\chi_\theta\|_{1,\Omega} \leq c_3L_1 + c_4L_2, \tag{5.67}$$

with

$$\begin{aligned}c_3 :=\ & 2\widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{\alpha}_{\mathrm{T}}^{-1}\bar{C}_{O_{\mathrm{T}}}[(C_\theta+1)C_{\mathrm{lift},2}\delta^{-3/4}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + (2C_2\gamma_{\mathbf{g}})^{-1}], \\ c_4 :=\ & 2\widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{\alpha}_{\mathrm{T}}^{-2}\widehat{C}_D\gamma_{\mathbf{g}}\bar{C}_{O_{\mathrm{T}}}(C_\theta+1)C_{\mathrm{lift},2}\delta^{-3/4}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \\ & + 2\widehat{\alpha}_{\mathrm{F}}^{-1}\widehat{\alpha}_{\mathrm{T}}^{-2}\widehat{C}_D\gamma_{\mathbf{g}}\bar{C}_{O_{\mathrm{T}}}(2C_2\gamma_{\mathbf{g}})^{-1} + \widehat{\alpha}_{\mathrm{T}}^{-1}.\end{aligned}$$

In turn, observing that estimate (5.37b) and the fact that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$ (cf. (5.39)) imply that $\mathbf{u}_h$ satisfies

$$\|\mathbf{u}_h\|_{1,\mathcal{T}_h^{\mathrm{f}}} \leq \|\mathbf{u}_h\|_{1,\mathbf{H}_h} \leq \widehat{C}_{\mathbf{u}}\gamma_{\mathbf{g}}\|\theta_{h,1}\|_{1,\Omega} \leq \widehat{C}_{\mathbf{u}}\gamma_{\mathbf{g}}\widehat{C}_{\mathrm{lift},2}\delta^{1/4}(2+\delta^{-1})\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}, \tag{5.68}$$

from (5.20), (5.57), (5.64), (5.68), and the fact that $\|\varrho_\theta\|_{1,\Omega} \le \|\varrho_\theta\|_{1,\Omega_{\rm f}} + c\|\varrho_\theta\|_{1,4,\Omega_{\rm m}}$, we deduce

$$\begin{aligned} L_1 &\le c_5\big(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathbf{H}_h} + \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\rm f}} + \|\varrho_\theta\|_{1,\Omega_{\rm f}} + \|\varrho_\theta\|_{1,4,\Omega_{\rm m}}\big), \\ L_2 &\le c_6\big(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathbf{H}_h} + \|\varrho_\theta\|_{1,\Omega_{\rm f}} + \|\varrho_\theta\|_{1,4,\Omega_{\rm m}}\big), \end{aligned} \tag{5.69}$$

with c_5, c_6 being positive constants independent of h.

Therefore, from (5.66), (5.67) and (5.69), the fact $\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathbf{H}_h} \le \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\rm f}} + \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{{\rm div};\Omega_{\rm m}}$, and the triangle inequality, it follows that there exists $c_7 > 0$, independent of h, such that

$$\|\mathbf{e}_{\mathbf{u}}\|_{1,\mathbf{H}_h} + \|{\rm e}_\theta\|_{1,\Omega} \le c_7\left(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\rm f}} + \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{{\rm div};\Omega_{\rm m}} + \|\varrho_\theta\|_{1,\Omega_{\rm f}} + \|\varrho_\theta\|_{1,4,\Omega_{\rm m}}\right), \tag{5.70}$$

which together with (5.55) implies (5.58).

On the other hand, to estimate ${\rm e}_p$ we first use the discrete inf-sup condition (5.35), to deduce that

$$\begin{aligned} \widehat{\beta}\|\chi_p\|_{0,\Omega} &\le \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},\chi_p)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} \\ &= \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},{\rm e}_p)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} + \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},-\varrho_p)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} \\ &\le \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},{\rm e}_p)}{\|\mathbf{v}\|_{1,\mathbf{H}_h}} + \|\varrho_p\|_{0,\Omega}. \end{aligned} \tag{5.71}$$

Then, noticing that adding and subtracting $O_{\rm F}^h(\mathbf{u}_h;\mathbf{u},\mathbf{v})$ in the first equation of (5.60), yields

$$B(\mathbf{v},{\rm e}_p) = -A_{\rm F}^h(\mathbf{e}_{\mathbf{u}},\mathbf{v}) - \big[O_{\rm F}^h(\mathbf{u};\mathbf{u},\mathbf{v}) - O_{\rm F}^h(\mathbf{u}_h;\mathbf{u},\mathbf{v})\big] - O_{\rm F}^h(\mathbf{u}_h;\mathbf{e}_{\mathbf{u}},\mathbf{v}) + D({\rm e}_\theta,\mathbf{v}),$$

for all $\mathbf{v} \in \mathbf{H}_h$, employing estimates (5.21b) (5.26) and (5.27) we obtain

$$\begin{aligned} |B(\mathbf{v},{\rm e}_p)| \le\ & \widehat{C}_{A_{\rm F}}\|\mathbf{e}_{\mathbf{u}}\|_{1,\mathbf{H}_h}\|\mathbf{v}\|_{1,\mathbf{H}_h} + C_{\rm Lip,F}\|\mathbf{e}_{\mathbf{u}}\|_{1,\mathcal{T}_h^{\rm f}}\|\mathbf{u}\|_{1,\mathcal{T}_h^{\rm f}}\|\mathbf{v}\|_{1,\mathbf{H}_h} \\ & + C_{\rm Lip,F}\|\mathbf{u}_h\|_{1,\mathcal{T}_h^{\rm f}}\|\mathbf{e}_{\mathbf{u}}\|_{1,\mathcal{T}_h^{\rm f}}\|\mathbf{v}\|_{1,\mathbf{H}_h} + \widehat{C}_D\gamma_{\mathbf{g}}\|{\rm e}_\theta\|_{1,\Omega}\|\mathbf{v}\|_{1,\mathbf{H}_h}, \end{aligned}$$

and then, from the latter, estimates (5.64), (5.68), (5.70), (5.71) and the triangle inequality, we obtain that there exists $c_8 > 0$, independent of h, such that

$$\begin{aligned} \|{\rm e}_p\|_{0,\Omega} &\le \|\chi_p\|_{0,\Omega} + \|\varrho_p\|_{0,\Omega} \\ &\le c_8\left(\|\boldsymbol{\varrho}_{\mathbf{u}}\|_{2,\mathcal{T}_h^{\rm f}} + \|\boldsymbol{\varrho}_{\mathbf{u}}\|_{{\rm div};\Omega_{\rm m}} + \|\varrho_\theta\|_{1,\Omega_{\rm f}} + \|\varrho_\theta\|_{1,4,\Omega_{\rm m}} + \|\varrho_p\|_{0,\Omega}\right), \end{aligned}$$

which together with (5.55), implies (5.59), and concludes the proof. □

5.5 Numerical results

In this section we restrict ourselves to the 2D case and we present some numerical results illustrating the performance of our nonconforming scheme (5.16) analyzed in Section 5.3 for approximating the solutions of (5.6), and to confirm the theoretical converges rates (5.58) and (5.59) predicted by the theory according to the Theorem 5.4.1. Our implementation is based on a *FreeFem++* code [68], in conjunction with the direct linear solver *UMFPACK* [41]. The experimental errors and convergence rates for the velocity, pressure, and temperature are the result of iterations based on a Picard-type algorithm over a family of quasi-uniform triangulations of the corresponding domains. This process ends when the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, that is

$$\frac{\|\mathbf{coeff}^{n+1} - \mathbf{coeff}^{n}\|_{l^2}}{\|\mathbf{coeff}^{n+1}\|_{l^2}} \le tol,$$

where tol is a fixed tolerance and $\|\cdot\|_{l^2}$ stands for the usual euclidean norm in $\mathbb{R}^{\mathrm{dof}}$, with dof denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$, Q_h and Ψ_h.

Now, we introduce some additional notations. As in Section 5.4, the individual errors for each variable are denoted by $\mathbf{e_u}$, e_p and e_θ. In addition, we define the experimental rates of convergence $\mathbf{r_u}$, r_p and r_θ, as

$$\mathbf{r_u} := \frac{\log(\mathbf{e_u}/\mathbf{e}'_{\mathbf{u}})}{\log(h/h')}, \quad \mathrm{r}_p := \frac{\log(\mathrm{e}_p/\mathrm{e}'_p)}{\log(h/h')} \quad \text{and} \quad \mathrm{r}_\theta := \frac{\log(\mathrm{e}_\theta/\mathrm{e}'_\theta)}{\log(h/h')},$$

where h and h' denote two consecutive mesh sizes with their respective errors $\mathbf{e}$, $\mathbf{e}'$ (or e, e$'$).

For all examples bellow, we simply take $\mathbf{u}^0 = \mathbf{0}$ and $\theta^0 = 0$ as initial guess, we consider $tol = 1e-6$ and $a^{\mathrm{pen}} = 5$.

Example 1: Manufactured exact solution

For our first example, we illustrate the robustness and accuracy of the method considering a manufactured exact solution defined on the following computational domain $\Omega = \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$, with $\Omega_{\mathrm{f}} := (-1,1)\times(0,1)$ and $\Omega_{\mathrm{m}} := (-1,1)\times(-1,0)$. We chose the parameters $\mu = 1$, $\mathbf{g}_{\mathrm{f}} = (0,-1)^t$, $\mathbf{g}_{\mathrm{m}} = (0,-1)^t$, $\omega_1 = 1$, $\kappa_{\mathrm{f}} = 1$, $\kappa_{\mathrm{m}} = 1$, $\mathbf{K} = \mathbf{I}$, $\boldsymbol{\kappa} = \mathbf{I}$, and the terms on the right-hand side are adjusted so that the exact solution is given by the functions:

$$\mathbf{u}(x,y) := \begin{pmatrix} -2\sin(\pi x)^2(y-1) \\ 2\pi\sin(\pi x)\cos(\pi x)(y-1)^2 \end{pmatrix}, \quad p(x,y) := x^5 + x^3 + xy,$$

$$\theta(x,y) := \exp(-xy),$$

We notice that $\mathbf{u}_{\mathrm{f}}|_\Sigma = \mathbf{u}_{\mathrm{m}}|_\Sigma$, $\theta_{\mathrm{f}}|_\Sigma = \theta_{\mathrm{m}}|_\Sigma$, and $\kappa_{\mathrm{f}} \nabla\theta_{\mathrm{f}}|_\Sigma = \kappa_{\mathrm{m}} \nabla\theta_{\mathrm{m}}|_\Sigma$. We notice also that these functions do not satisfy the interface conditions (5.3), thus the difference must be incorporated as a functional at the right-hand side of the resulting system.

In Table 5.1 we summarize the convergence history on a sequence of quasi-uniform triangulations for the finite element families presented in (5.14) with $k = 1$. We observe there that the rate of convergence $O(h)$ predicted by Theorem 5.4.1 is attained in all the cases for all unknowns. In addition, in the last two columns we show the l^∞-norm of $\operatorname{div} \mathbf{u}_h$ and the number of iterations required to stop the algorithm, respectively. In particular, we observe there that the velocity is practically divergence-free for all refinement steps.

dof	h	$\mathbf{e}_\mathbf{u}$	$\mathbf{r}_\mathbf{u}$	e_p	r_p	e_θ	r_θ	$\\|\operatorname{div} \mathbf{u}_h\\|_{l^\infty}$	itt
2816	0.190	4.178	–	2.143	–	0.162	–	4.263e-14	10
10517	0.097	1.912	1.175	0.941	1.237	0.084	0.989	8.526e-14	10
41201	0.051	0.915	1.158	0.462	1.119	0.042	1.067	2.8421-13	10
164714	0.027	0.437	1.154	0.232	1.076	0.021	1.078	6.821e-13	10
672281	0.014	0.216	1.167	0.126	1.011	0.010	1.183	1.364e-12	10

Table 5.1: EXAMPLE 1: Degrees of Freedom, mesh sizes, errors, rates of convergence, l^∞-norm of $\operatorname{div} \mathbf{u}_h$ and number of iterations for (5.16).

Example 2: Kovasznay's analytical solution

In our second example, we focus on the performance of the iterative method with respect to the viscosity μ. To this end, we consider the domain $\Omega = \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$, with $\Omega_{\mathrm{f}} := (-1/2, 3/2) \times (0, 1/2)$ and $\Omega_{\mathrm{m}} := (-1/2, 3/2) \times (-1/2, 0)$. In turn, the terms on the right-hand side are adjusted so that the exact solution are given by the functions

$$\mathbf{u}(x, y) := \begin{pmatrix} 1 - e^{\lambda x} \cos(2\pi y) \\ \frac{\lambda}{2\pi} e^{\lambda x} \sin(2\pi y) \end{pmatrix} \quad \text{in} \quad \Omega,$$

$$p(x, y) := \tfrac{-1}{2} e^{2\lambda x} + c_0 \quad \text{in} \quad \Omega$$

$$\theta(x, y) := \exp(-xy) \quad \text{in} \quad \Omega.$$

Above, λ is given by

$$\lambda := \frac{-8\pi^2}{\mu^{-1} + \sqrt{\mu^{-2} + 16\pi^2}},$$

with $\mu > 0$ being the viscosity of the fluid and c_0 is a constant chosen in such a way $(p, 1)_\Omega = 0$. We note that $(\mathbf{u}_{\mathrm{f}}, p_{\mathrm{f}})$ is the well known analytical solution for the

Navier–Stokes problem obtained by Kovasznay in [73], which presents a boundary layer at $\{-1/2\} \times (0, 2)$.

In Table 5.2 we show the behavior of the iterative method as a function of the viscosity μ, considering different mesh sizes h. As expected, when viscosity is reduced, the method requires a larger number of iterations to achieve convergence. We do not present numerical experiments for smaller values of μ as, in such instances, the maximum iteration limit set in the code (200 iterations) is reached for all meshes.

μ	$h = 0.362$	$h = 0.180$	$h = 0.094$	$h = 0.047$	$h = 0.023$	$h = 0.013$
1	9	9	9	9	9	9
0.1	13	14	14	14	14	14
0.01	14	17	17	17	17	17
0.001	**	31	31	31	31	31

Table 5.2: Example 2: Number of iterations of the iterative method with respect to μ.

Example 3: Dimensionless problem

In our last example, we consider the domain $\Omega = \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$, with $\Omega_{\mathrm{f}} := (-1, 1) \times (0, d_{\mathrm{f}})$ and $\Omega_{\mathrm{m}} := (-1, 1) \times (-d_{\mathrm{m}}, 0)$ and address the behavior of convection patterns as d_{f} decreases. As documented in [79] and [80], it has been observed that when the free-fluid region has a shallow depth, the convection patterns extend to encompass both domains (full convection). Conversely, when the free-fluid region has greater depth, the dominant convection patterns are mainly localized within the free-fluid region (fluid-dominated convection). To study this behavior, we proceed similarly to [64, 79, 80] and consider the following dimensionless problem:

$$
\begin{aligned}
&\widetilde{\boldsymbol{\sigma}}_{\mathrm{f}} = 2\,\mathbf{e}(\widetilde{\mathbf{u}}_{\mathrm{f}}) - \widetilde{p}_{\mathrm{f}}\,\mathbf{I} \quad \text{in} \quad \Omega_{\mathrm{f}},\\
&-\mathbf{div}(\,\widetilde{\boldsymbol{\sigma}}_{\mathrm{f}}) + (\widetilde{\mathbf{u}}_{\mathrm{f}}\cdot\nabla)\,\widetilde{\mathbf{u}}_{\mathrm{f}} + \mathrm{Ra}_{\mathrm{f}}\widetilde{\theta}_{\mathrm{f}}\mathbf{g}_{\mathrm{f}} = \mathbf{0} \quad \text{in} \quad \Omega_{\mathrm{f}},\\
&\mathrm{div}\,\widetilde{\mathbf{u}}_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}},\\
&-\epsilon_T\Delta\widetilde{\theta}_{\mathrm{f}} + \mathrm{Pr}_{\mathrm{f}}\,\widetilde{\mathbf{u}}_{\mathrm{f}}\cdot\nabla\,\widetilde{\theta}_{\mathrm{f}} + \tfrac{1}{\epsilon_T}\widetilde{\mathbf{u}}_{\mathrm{f}}\cdot\mathbf{g}_{\mathrm{f}} = 0 \quad \text{in} \quad \Omega_{\mathrm{f}},\\
&\tfrac{1}{\mathrm{Da}}\widetilde{\mathbf{u}}_{\mathrm{m}} + \nabla\widetilde{p}_{\mathrm{m}} = -\tfrac{\mathrm{Ra}_{\mathrm{m}}}{\mathrm{Da}}\widetilde{\theta}_{\mathrm{m}}\mathbf{g}_{\mathrm{m}} \quad \text{in} \quad \Omega_{\mathrm{m}},\\
&\mathrm{div}\,\widetilde{\mathbf{u}}_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}},\\
&-\Delta\widetilde{\theta}_{\mathrm{m}} + \mathrm{Pr}_{\mathrm{m}}\,\widetilde{\mathbf{u}}_{\mathrm{m}}\cdot\nabla\widetilde{\theta}_{\mathrm{m}} = -\widetilde{\mathbf{u}}_{\mathrm{m}}\cdot\mathbf{g}_{\mathrm{m}} \quad \text{in} \quad \Omega_{\mathrm{m}},\\
&\widetilde{\theta}_{\mathrm{f}} = \widetilde{\theta}_{\mathrm{m}} \quad \text{on} \quad \Sigma,\\
&\epsilon_T\nabla\widetilde{\theta}_{\mathrm{f}}\cdot\mathbf{n} = \nabla\widetilde{\theta}_{\mathrm{m}}\cdot\mathbf{n} \quad \text{on} \quad \Sigma,\\
&\widetilde{\mathbf{u}}_{\mathrm{f}}\cdot\mathbf{n} = \widetilde{\mathbf{u}}_{\mathrm{m}}\cdot\mathbf{n} \quad \text{on} \quad \Sigma,\\
&\widetilde{\boldsymbol{\sigma}}_{\mathrm{f}}\,\mathbf{n} + \tfrac{\omega_1}{\sqrt{\mathrm{Da}}}(\widetilde{\mathbf{u}}_{\mathrm{f}}\cdot\mathbf{t})\,\mathbf{t} = -\widetilde{p}_{\mathrm{m}}\,\mathbf{n} \quad \text{on} \quad \Sigma,\\
&\widetilde{\theta}_{\mathrm{f}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{f}},\\
&\widetilde{\mathbf{u}}_{\mathrm{f}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{f}},\\
&\widetilde{\theta}_{\mathrm{m}} = 0 \quad \text{on} \quad \Gamma_{\mathrm{m}},\\
&\widetilde{\mathbf{u}}_{\mathrm{m}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{m}},
\end{aligned}
\tag{5.72}
$$

where $\mathrm{Pr}_\star$ and $\mathrm{Ra}_\star$ represent the Prandtl and Rayleigh numbers in the domain $\Omega_\star$ for $\star \in \{\mathrm{f},\mathrm{m}\}$, Da represents the Darcy number, and $\epsilon_T = \frac{\kappa_{\mathrm{f}}}{\kappa_{\mathrm{m}}}$ is the ratio of thermal diffusivities. Additionally, we define the dimensionless number $\widehat{d} = \frac{d_{\mathrm{f}}}{d_{\mathrm{m}}}$ which represents the depth ratio.

We observe that the temperature profile can be recovered by means of the relationship

$$\theta = (\overline{T}_{\mathrm{f}} + \widetilde{\theta}_{\mathrm{f}}(T_U - T_0)\mathrm{Pr}_{\mathrm{f}}\epsilon_T^{-1})\chi_{\mathrm{f}} + (\overline{T}_{\mathrm{m}} + \widetilde{\theta}_{\mathrm{m}}(T_0 - T_B)\mathrm{Pr}_{\mathrm{m}})\chi_{\mathrm{m}},$$

where $\widetilde{\theta}_\star$ for $\star \in \{\mathrm{f},\mathrm{m}\}$ is the temperature solution of (5.72), T_U and T_B are constant temperatures in the upper and bottom boundaries, respectively,

$$\overline{T}_{\mathrm{f}} := T_0 + y\frac{T_U - T_0}{d_{\mathrm{f}}}, \qquad \overline{T}_{\mathrm{m}} := T_0 + y\frac{T_0 - T_B}{d_{\mathrm{m}}} \quad \text{and} \quad \overline{T}_0 := \frac{\widehat{d}T_B + \epsilon_T T_U}{\widehat{d} + \epsilon_T}$$

In [79] and [80] is also established that if $T_U > T_B$, the conductive state is stable, whereas if $T_B > T_U$, buoyancy can destabilize the system.

In Figure 5.2, we present flow configurations (streamlines) and temperature profiles (color) for two cases with different $\widehat{d}$ values, specifically $\widehat{d} = 0.2$ and $\widehat{d} =$

0.35. We set the temperatures as $T_B = 1$ and $T_U = 0$, representing an unstable scenario. The Figure on the bottom, similarly to [79] and [80], we observe a situation of full convection with the following parameter values: $d_\mathrm{f} = 0.2$, $d_\mathrm{m} = 1.0$, and $\mathrm{Ra}_\mathrm{m} = 30$, while the figure on the top, we witness a fluid-dominated convection scenario characterized by the parameters $d_\mathrm{f} = 0.35$, $d_\mathrm{m} = 1.0$, and $\mathrm{Ra}_\mathrm{m} = 10$. For both cases, we consider the following parameter settings: $\mathrm{Pr}_\mathrm{f} = 0.7$, $\mathrm{Pr}_\mathrm{m} = 0.7$, $\epsilon_T = 0.7$, $\mathrm{Da} = 1.0 \times 10^{-4}$, $\alpha_d = 1.0$, $\mathbf{g}_\mathrm{f} = (0,1)^t$, $\mathbf{g}_\mathrm{m} = (0,1)^t$, and the Rayleigh number within Ω_f is calculated as $\mathrm{Ra}_\mathrm{f} = \frac{\mathrm{Ra}_\mathrm{m}}{\mathrm{Da}}$.

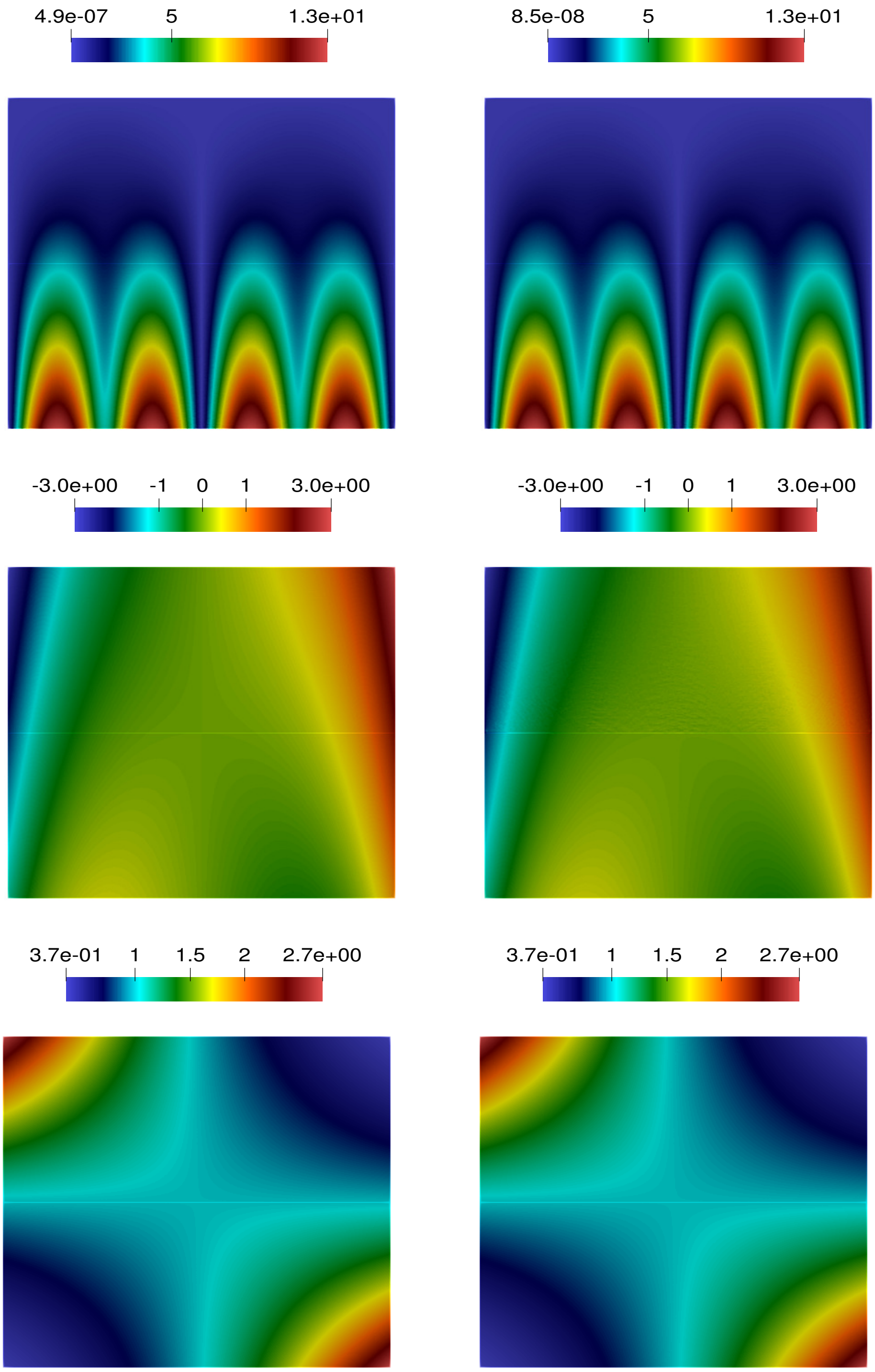

Figure 5.1: Example 1: On left: exact magnitude of the velocity (left-top), pressure (left-center) and temperature (left-bottom). On right: approximate magnitude of the velocity (right-top), pressure (right-center) and temperature (right-bottom).

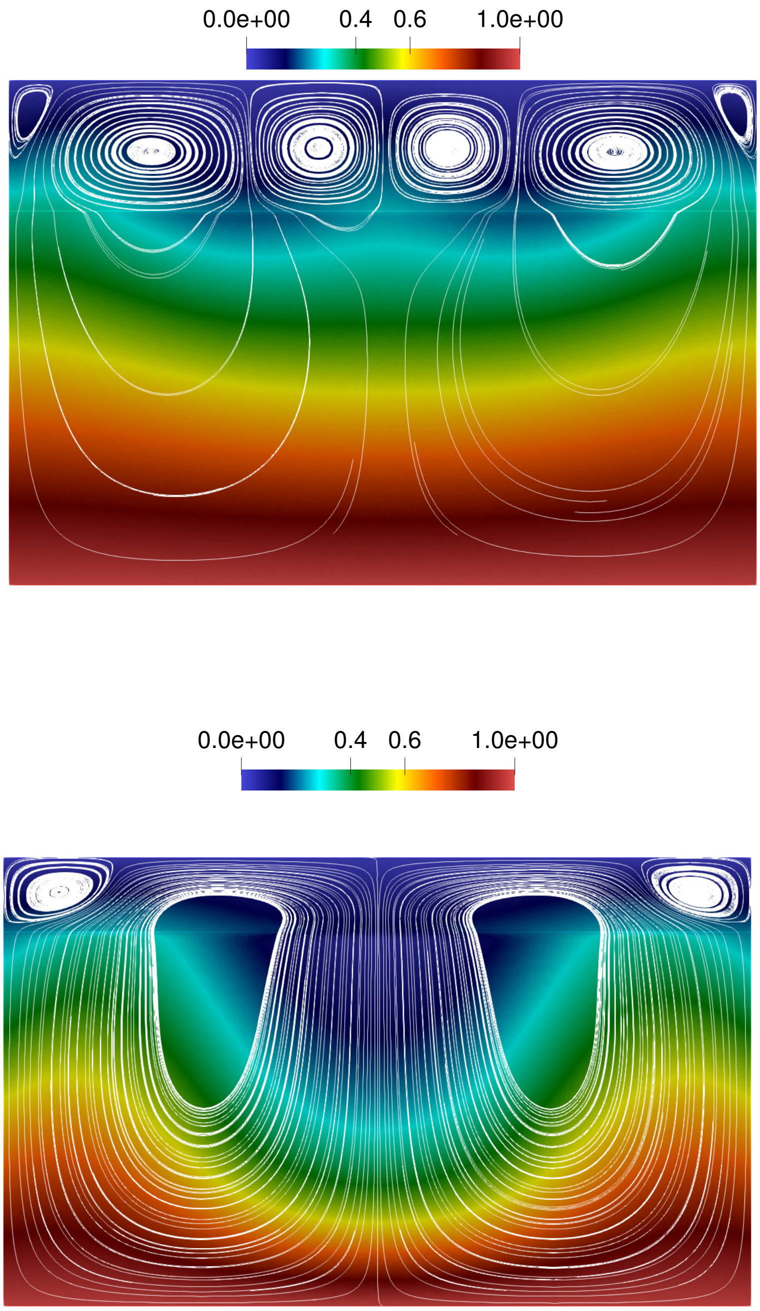

Figure 5.2: EXAMPLE 3: Fluid-dominated and Full convection with temperature (colour) and streamlines (contour), with $\widehat{d} = 0.35$, $\mathrm{Ra_m} = 10$ (top) and $\widehat{d} = 0.2$, $\mathrm{Ra_m} = 30$ (bottom). Fixed parameters $\mathrm{Pr_f} = 0.7$, $\mathrm{Pr_m} = 0.7$, $\epsilon_T = 0.7$, $\mathrm{Da} = 1.0 \times 10^{-4}$, and $\alpha_d = 1.0$.

Chapter 6

Conclusions and future work

6.1 Concluding remarks

The main contribution of this thesis has been to propose and analyze conforming and non-conforming numerical schemes for a non-isothermal fluid-membrane model present in desalination processes.

In **Chapter 2**, the model problem was presented, consisting of two regions: one occupied by a free-fluid and the other by a porous membrane. The governing equations for the system were formulated as a coupling between the Navier–Stokes and heat equations in the free-fluid region and the Darcy and heat equations within the membrane.

In **Chapter 3**, we established a conforming finite element method for the non-isothermal fluid-membrane interaction problem introduced in Chapter 2, which, to the best of our knowledge, represents the first contribution in this direction. Specifically, we employed a velocity-pressure-temperature variational formulation for the Boussinesq system and a dual-mixed scheme coupled with a primal formulation for the Darcy and Heat equations in the membrane region. This approach involves the introduction of the pressure of the porous medium as an additional unknown at the common interface via a Lagrange multiplier. We established the existence of a solution by combining the Galerkin method with a fixed-point strategy, and for the uniqueness of the solution to the coupled problem, it was necessary to assume a more restrictive condition on the data and the temperature of the porous medium. We proposed a conforming scheme to approximate the solution of the coupled problem, employing Bernardi–Raugel and Raviart–Thomas elements for velocities, piecewise constant elements for pressures, continuous piecewise linear functions for temperatures, and continuous piecewise linear functions for the Lagrange multiplier on a partition of the interface. In addition, we derived the corresponding error estimates and performed several numerical simulations based on manufactured solutions, validating the effectiveness and accuracy of our proposed method.

In **Chapter 4**, we developed a residual-based a posteriori error estimator for the conforming discretization of the nonisothermal Navier–Stokes/Darcy coupled system introduced in Chapter 3. Using standard techniques such as global inf-sup conditions, appropriate Helmholtz's decompositions, and the local approximation properties of the Raviart–Thomas and Clément interpolation operators, we derived an a posteriori error estimator and proved its reliability. Additionally, we employed inverse inequalities, localization techniques of bubble functions, and known results from previous works to prove the local efficiency of the proposed error estimator. Finally, we developed an adaptive algorithm based on the local and computable error indicators provided by the a posteriori error estimator.

In **Chapter 5**, we proposed and analyzed an $\mathbf{H}(\mathrm{div})$-conforming and mass-conservative finite element method for the nonisothermal fluid-membrane interaction problem introduced in Chapter 2. This approach preserve the continuity of the normal components of the velocity exactly along the interface, eliminating the need to introduce the Lagrange multiplier as was done in Chapter 3. As a result, the unknowns in the formulation were reduced to the velocity, pressure, and temperature in both domains. For the associated Galerkin scheme, we combined an $\mathbf{H}(\mathrm{div})$-conforming scheme for the fluid variables with a conforming Galerkin discretization for the heat equation. The analysis of the discrete scheme was performed without modifying the convective term of the heat equation and, to enforce the H^1 continuity of the velocity in the free-fluid region, we utilized an interior penalty discontinuous Galerkin technique. The resulting numerical scheme produced exactly divergence-free velocities and preserved the law of conservation of mass at a discrete level. Furthermore, we proved well-posedness for both the continuous and discrete schemes under the assumption of sufficiently small data and employing a fixed-point strategy. Subsequently, by imposing conditions on the data and the temperature in the membrane, we have derived the theoretical rate of convergence. We performed several numerical simulations based on manufactured solutions to validate the effectiveness and accuracy of our proposed method and the robustness of the discontinuous scheme.

6.2 Future works

The development of this thesis and the results obtained have motivated us to develop new works and extend the existing literature on phenomena of this type. Some of them are detailed below:

6.2.1 A fully discontinuous finite element method for a non-isothermal fluid-membrane interaction

The first objective that we propose after the development of this thesis is to propose a completely discontinuous finite element method different from the one developed in Chapter 5. This approach will allow us to establish a discrete scheme that does not require the introduction of any extension operator for the Dirichlet temperature boundary condition, thereby achieving a well-posed formulation with a less restrictive assumptions.

6.2.2 A hybridizable discontinuous Galerkin method for a nonisothermal fluid-membrane interaction

The next objective that we propose is to extend the analysis developed in Chapter 5 employing a Hybridizable Discontinuous Galerkin method for the non-isothermal fluid-membrane interaction established in Chapter 2, in order to reduce the computational cost. In particular, we are interested in proposing an alternative discrete scheme that preserves divergence-free velocities at the discrete level and guarantees pressure robustness.

6.2.3 A mixed finite element method for a nonisothermal fluid-membrane interaction

Another objective we intend to pursue after this thesis is to propose a mixed finite element method for the set of equations described in Chapter 2. More precisely, we are interested in considering a mixed formulation for the heat equation. By doing that, we can approximate directly additional variables of interest, such as the velocity gradient as well as the heat-flux.

6.2.4 A time-dependent finite element method for a non-isothermal fluid-membrane interaction

For the final proposed future work, we are interesting in develop an analysis for the temporal version of the problem discussed in Chapter 2. Specifically, we are interested in investigating a problem similar to the one previously mentioned, but incorporating the time derivative of the velocity into our equations.

Bibliography

[1] S. F. Anis, R. Hashaikeh, and N. Hilal. Reverse osmosis pretreatment technologies and future trends: A comprehensive review. *Desalination*, 452:159–195, 2019.

[2] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.

[3] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2001/02.

[4] I. Babuška and G. N. Gatica. A residual-based a posteriori error estimator for the Stokes-Darcy coupled problem. *SIAM J. Numer. Anal.*, 48(2):498–523, 2010.

[5] L. Badea, M. Discacciati, and A. Quarteroni. Numerical analysis of the Navier-Stokes/Darcy coupling. *Numer. Math.*, 115(2):195–227, 2010.

[6] G. Beavers and D. Joseph. Boundary conditions at a naturally permeable wall. *Journal of Fluid Mechanics*, 30:197–207, 10 1967.

[7] C. Bernardi, S. Dib, V. Girault, F. Hecht, F. Murat, and T. Sayah. Finite element methods for Darcy's problem coupled with the heat equation. *Numer. Math.*, 139(2):315–348, 2018.

[8] C. Bernardi, S. Maarouf, and D. Yakoubi. Spectral discretization of Darcy's equations coupled with the heat equation. *IMA J. Numer. Anal.*, 36(3):1193–1216, 2016.

[9] C. Bernardi, B. Métivet, and B. Pernaud-Thomas. Couplage des équations de Navier-Stokes et de la chaleur: le modèle et son approximation par éléments finis. *RAIRO Modél. Math. Anal. Numér.*, 29(7):871–921, 1995.

[10] C. Bernardi, B. Métivet, and B. Pernaud-Thomas. Couplage des équations de Navier-Stokes et de la chaleur: le modèle et son approximation par éléments finis. *RAIRO Modél. Math. Anal. Numér.*, 29(7):871–921, 1995.

[11] C. Bernardi and G. Raugel. Analysis of some finite elements for the Stokes problem. *Math. Comp.*, 44(169):71–79, 1985.

[12] D. Boffi, F. Brezzi, and M. Fortin. *Mixed finite element methods and applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2013.

[13] J. Boland and W. Layton. An analysis of the finite element method for natural convection problems. *Numer. Methods Partial Differential Equations*, 6(2):115–126, 1990.

[14] J. Boussinesq. *Théorie de l'écoulement tourbillonnant et tumultueux des liquides dans les lits rectilignes a grande section.* 2018.

[15] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[16] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[17] F. Brezzi, J. Douglas, Jr., and L. D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numer. Math.*, 47(2):217–235, 1985.

[18] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[19] J. Camaño, R. Oyarzúa, M. Serón, and M. Solano. A mass conservative finite element method for a nonisothermal Navier–Stokes/Darcy coupled system. Preprint on webpage at `https://www.ci2ma.udec.cl/publicaciones/prepublicaciones/prepublicacion.php?id=536`.

[20] J. Camaño, R. Oyarzúa, M. Serón, and M. Solano. A conforming finite element method for a nonisothermal fluid-membrane interaction. *Math. Comp.*, In press, 2025.

[21] J. Camaño, R. Oyarzúa, M. Serón, and M. Solano. A strong mass conservative finite element method for the navier–stokes/darcy coupled system. *Appl. Math. Lett.*, 163:109447, 2025.

[22] C. Carstensen. An a posteriori error estimate for a first-kind integral equation. *Math. Comp.*, 66(217):139–155, 1997.

[23] C. Carstensen. A posteriori error estimate for the mixed finite element method. *Math. Comp.*, 66(218):465–476, 1997.

[24] S. Caucao, G. N. Gatica, and R. Oyarzúa. A posteriori error analysis of a fully-mixed formulation for the Navier-Stokes/Darcy coupled problem with nonlinear viscosity. *Comput. Methods Appl. Mech. Engrg.*, 315:943–971, 2017.

[25] S. Caucao, R. Oyarzúa, and S. Villa-Fuentes. A new mixed-FEM for steady-state natural convection models allowing conservation of momentum and thermal energy. *Calcolo*, 57(4):Paper No. 36, 39, 2020.

[26] S. Caucao, R. Oyarzúa, and S. Villa-Fuentes. A posteriori error analysis of a momentum and thermal energy conservative mixed FEM for the Boussinesq equations. *Calcolo*, 59(4):Paper No. 45, 40, 2022.

[27] A. Çeşmelioğlu and B. Rivière. Analysis of time-dependent Navier-Stokes flow coupled with Darcy flow. *J. Numer. Math.*, 16(4):249–280, 2008.

[28] A. Cesmelioglu and S. Rhebergen. A hybridizable discontinuous Galerkin method for the coupled Navier-Stokes and Darcy problem. *J. Comput. Appl. Math.*, 422:Paper No. 114923, 17, 2023.

[29] P. Chidyagwai and B. Rivière. On the solution of the coupled Navier-Stokes and Darcy equations. *Comput. Methods Appl. Mech. Engrg.*, 198(47-48):3806–3820, 2009.

[30] P. G. Ciarlet. *Linear and nonlinear functional analysis with applications*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[31] P. Clément. Approximation by finite element functions using local regularization. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.*, 9(R-2):77–84, 1975.

[32] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47(2):1319–1365, 2009.

[33] B. Cockburn, G. Kanschat, and D. Schotzau. A locally conservative LDG method for the incompressible Navier-Stokes equations. *Math. Comp.*, 74(251):1067–1095, 2005.

[34] B. Cockburn, G. Kanschat, and D. Schötzau. A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations. *J. Sci. Comput.*, 31(1-2):61–73, 2007.

[35] E. Colmenares, G. N. Gatica, and R. Oyarzúa. Analysis of an augmented mixed-primal formulation for the stationary Boussinesq problem. *Numer. Methods Partial Differential Equations*, 32(2):445–478, 2016.

[36] E. Colmenares, G. N. Gatica, and R. Oyarzúa. A posteriori error analysis of an augmented mixed-primal formulation for the stationary Boussinesq model. *Calcolo*, 54(3):1055–1095, 2017.

[37] E. Colmenares and M. Neilan. Dual-mixed finite element methods for the stationary Boussinesq problem. *Comput. Math. Appl.*, 72(7):1828–1850, 2016.

[38] P. Cornel and S. Krause. *Membrane Bioreactors for Wastewater Treatment*, chapter 9, pages 217–238. John Wiley & Sons, Ltd, 2008.

[39] R. Courant. Variational methods for the solution of problems of equilibrium and vibrations. *Bull. Amer. Math. Soc.*, 49:1–23, 1943.

[40] A. Dalal and M. K. Das. Natural convection in a rectangular cavity heated from below and uniformly cooled from the top and both sides. *Numerical Heat Transfer, Part A: Applications*, 49(3):301–322, 2006.

[41] T. Davis. Algorithm 832: Umfpack v4.3—an unsymmetric-pattern multifrontal method. *ACM Trans. Math. Softw.*, 30:196–199, 01 2004.

[42] D. A. Di Pietro and A. Ern. *Mathematical aspects of discontinuous Galerkin methods*, volume 69 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Heidelberg, 2012.

[43] S. Dib, V. Girault, F. Hecht, and T. Sayah. *A posteriori* error estimates for Darcy's problem coupled with the heat equation. *ESAIM Math. Model. Numer. Anal.*, 53(6):2121–2159, 2019.

[44] M. Discacciati. Coupling free and porous-media flows: models and numerical approximation. In *Simulation of flow in porous media*, volume 12 of *Radon Ser. Comput. Appl. Math.*, pages 107–138. De Gruyter, Berlin, 2013.

[45] M. Discacciati and R. Oyarzúa. A conforming mixed finite element method for the Navier-Stokes/Darcy coupled problem. *Numer. Math.*, 135(2):571–606, 2017.

[46] E. Drioli, Y. Wu, and V. Calabro. Membrane distillataion in the treatment of aqueous solutions. *Journal of Membrane Science*, 33(3):277–284, 1987.

[47] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[48] M. Farhloul, S. Nicaise, and L. Paquet. A mixed formulation of Boussinesq equations: analysis of nonsingular solutions. *Math. Comp.*, 69(231):965–986, 2000.

[49] M. Farhloul, S. Nicaise, and L. Paquet. A refined mixed finite element method for the Boussinesq equations in polygonal domains. *IMA J. Numer. Anal.*, 21(2):525–551, 2001.

[50] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Linearized steady problems.

[51] G. N. Gatica. A note on the efficiency of residual-based a-posteriori error estimators for some mixed finite element methods. *Electron. Trans. Numer. Anal.*, 17:218–233, 2004.

[52] G. N. Gatica. *A simple introduction to the mixed finite element method*. SpringerBriefs in Mathematics. Springer, Cham, 2014. Theory and applications.

[53] G. N. Gatica. A note on stable Helmholtz decompositions in 3D. *Appl. Anal.*, 99(7):1110–1121, 2020.

[54] G. N. Gatica, G. C. Hsiao, and S. Meddahi. A residual-based a posteriori error estimator for a two-dimensional fluid-solid interaction problem. *Numer. Math.*, 114(1):63–106, 2009.

[55] G. N. Gatica, G. C. Hsiao, and S. Meddahi. A coupled mixed finite element method for the interaction problem between an electromagnetic field and an elastic body. *SIAM J. Numer. Anal.*, 48(4):1338–1368, 2010.

[56] G. N. Gatica, S. Meddahi, and R. Oyarzúa. A conforming mixed finite-element method for the coupling of fluid flow with porous media flow. *IMA J. Numer. Anal.*, 29(1):86–108, 2009.

[57] G. N. Gatica, S. Meddahi, and R. Ruiz-Baier. An L^p spaces-based formulation yielding a new fully mixed finite element method for the coupled Darcy and heat equations. *IMA J. Numer. Anal.*, 42(4):3154–3206, 2022.

[58] G. N. Gatica, R. Oyarzúa, and F.-J. Sayas. Analysis of fully-mixed finite element methods for the Stokes-Darcy coupled problem. *Math. Comp.*, 80(276):1911–1948, 2011.

[59] G. N. Gatica, R. Oyarzúa, and F.-J. Sayas. A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes-Darcy coupled problem. *Comput. Methods Appl. Mech. Engrg.*, 200(21-22):1877–1891, 2011.

[60] G. N. Gatica, R. Oyarzúa, and N. Valenzuela. A five-field augmented fully-mixed finite element method for the Navier-Stokes/Darcy coupled problem. *Comput. Math. Appl.*, 80(8):1944–1963, 2020.

[61] V. Girault, G. Kanschat, and B. Rivière. On the coupling of incompressible Stokes or Navier-Stokes and Darcy flow through porous media. In *Modelling and simulation in fluid dynamics in porous media*, volume 28 of *Springer Proc. Math. Stat.*, pages 1–25. Springer, New York, 2013.

[62] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[63] V. Girault and B. Rivière. DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition. *SIAM J. Numer. Anal.*, 47(3):2052–2089, 2009.

[64] D. Gobin and G. Benoit. Natural convection in partially porous media: A brief overview. *International Journal of Numerical Methods for Heat & Fluid Flow*, 18:465–490, 05 2008.

[65] C. Gostoli, G. C. Sarti, and S. Matulli. Low temperature distillation through hydrophobic membranes. *Separation Science and Technology*, 22(2-3):855–872, 1987.

[66] L. F. Greenlee, D. F. Lawler, B. D. Freeman, B. Marrot, and P. Moulin. Reverse osmosis desalination: Water sources, technology, and today's challenges. *Water Research*, 43(9):2317–2348, 2009.

[67] H. Hayer, O. Bakhtiari, and T. Mohammadi. Simulation of momentum, heat and mass transfer in direct contact membrane distillation: A computational fluid dynamics approach. *Journal of Industrial and Engineering Chemistry*, 21:1379–1382, 2015.

[68] F. Hecht. New development in freefem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.

[69] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, 11:237–339, 2002.

[70] G. Kanschat and B. Rivière. A strongly conservative finite element method for the coupling of Stokes and Darcy flow. *J. Comput. Phys.*, 229(17):5933–5943, 2010.

[71] O. A. Karakashian and W. N. Jureidini. A nonconforming finite element method for the stationary Navier-Stokes equations. *SIAM J. Numer. Anal.*, 35(1):93–120, 1998.

[72] S. Kimura, S.-I. Nakao, and S.-I. Shimatani. Transport phenomena in membrane distillation. *Journal of Membrane Science*, 33(3):285–298, 1987.

[73] L. Kovasznay. Laminar flow behind two-dimensional grid. *Mathematical Proceedings of the Cambridge Philosophical Society*, 44:58 – 62, 01 1948.

[74] F. Laganà, G. Barbieri, and E. Drioli. Direct contact membrane distillation: modelling and concentration experiments. *Journal of Membrane Science*, 166(1):1–11, 2000.

[75] P. Lesaint and P.-A. Raviart. On a finite element method for solving the neutron transport equation. pages 89–123, 1974.

[76] S. Liu, L. Jiang, K. L. Chong, X. Zhu, Z.-H. Wan, R. Verzicco, R. J. A. M. Stevens, D. Lohse, and C. Sun. From rayleigh–bénard convection to porous-media convection: how porosity affects heat transfer and flow structure. *Journal of Fluid Mechanics*, 895:A18, 2020.

[77] T. Luo, R. Young, and P. Reig. Aqueduct projected water stress country rankings. *World Resources Institute, Washington, D. C.*, 2015.

[78] N. Mancosu, R. L. Snyder, G. Kyriakakis, and D. Spano. Water scarcity and future challenges for food production. *Water*, 7(3):975–992, 2015.

[79] M. McCurdy, N. Moore, and X. Wang. Convection in a coupled free flow-porous media system. *SIAM J. Appl. Math.*, 79(6):2313–2339, 2019.

[80] M. McCurdy, N. Moore, and X. Wang. Predicting convection configurations in coupled fluid–porous systems. *Journal of Fluid Mechanics*, 953, 12 2022.

[81] A. Mikelic and W. Jäger. On the interface boundary condition of beavers, joseph, and saffman. *SIAM Journal on Applied Mathematics*, 60(4):1111–1127, 2000.

[82] M. Nair and D. Kumar. Water desalination and challenges: The middle east perspective: A review. *Desalination and Water Treatment*, 51:2030–2040, 02 2013.

[83] R. Oyarzúa, T. Qin, and D. Schötzau. An exactly divergence-free finite element method for a generalized Boussinesq problem. *IMA J. Numer. Anal.*, 34(3):1104–1135, 2014.

[84] R. Oyarzúa and M. Serón. A divergence-conforming DG-mixed finite element method for the stationary Boussinesq problem. *J. Sci. Comput.*, 85(1):Paper No. 14, 36, 2020.

[85] R. Oyarzúa and P. Zúñiga. Analysis of a conforming finite element method for the Boussinesq problem with temperature-dependent parameters. *J. Comput. Appl. Math.*, 323:71–94, 2017.

[86] A. Parasyris, C. Brady, D. Das, and M. Discacciati. Computational modeling of coupled free and porous media flow for membrane-based filtration systems: A review. *Journal of Applied Membrane Science & Technology*, 23, 09 2019.

[87] V. Perfilov, A. Ali, and V. Fila. A general predictive model for direct contact membrane distillation. *Desalination*, 445, 11 2018.

[88] V. Perfilov, A. Ali, and V. Fila. A general predictive model for direct contact membrane distillation. *Desalination*, 445:181–196, 2018.

[89] A. Quarteroni and A. Valli. *Numerical approximation of partial differential equations*, volume 23 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1994.

[90] A. Quarteroni and A. Valli. *Domain decomposition methods for partial differential equations*. Numerical Mathematics and Scientific Computation. The Clarendon Press, Oxford University Press, New York, 1999. Oxford Science Publications.

[91] P.-A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, Lecture Notes in Math., Vol. 606, pages 292–315. Springer, Berlin, 1977.

[92] W. H. Reed and T. R. Hill. Triangular mesh methods for the neutron transport equation. *Technical Report LA-UR-73-0479*, 1973.

[93] B. Rivière and I. Yotov. Locally conservative coupling of Stokes and Darcy flows. *SIAM J. Numer. Anal.*, 42(5):1959–1977, 2005.

[94] P. Saffman. On the boundary condition at the surface of porous medium. *Studies in Applied Mathematics*, 50, 06 1971.

[95] R. Schofield, A. Fane, and C. Fell. Heat and mass transfer in membrane distillation. *Journal of Membrane Science*, 33(3):299–313, 1987.

[96] D. Schötzau, C. Schwab, and A. Toselli. Mixed *hp*-DGFEM for incompressible flows. *SIAM J. Numer. Anal.*, 40(6):2171–2194 (2003), 2002.

[97] M. M. A. Shirazi, A. Kargari, A. F. Ismail, and T. Matsuura. Computational fluid dynamic (cfd) opportunities applied to the membrane distillation process: State-of-the-art and perspectives. *Desalination*, 377:73–90, 2016.

[98] C. Skuse, A. Gallego-Schmid, A. Azapagic, and P. Gorgojo. Can emerging membrane-based desalination technologies replace reverse osmosis? *Desalination*, 500:114844, 2021.

[99] R. Temam. *Navier-Stokes equations. Theory and numerical analysis.* Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[100] R. Verfürth. A posteriori error estimation and adaptive mesh-refinement techniques. *J. Comput. Appl. Math.*, 50(1-3):67–83, 1994.

[101] J. P. Villaluenga and Y. Cohen. Numerical model of non-isothermal pervaporation in a rectangular channel. *Journal of Membrane Science*, 260(1):119–130, 2005.

[102] X. Yang, H. Yu, R. Wang, and A. G. Fane. Optimization of microstructured hollow fiber design for membrane distillation applications using cfd modeling. *Journal of Membrane Science*, 421-422:258–270, 2012.

[103] H. Yu, X. Yang, R. Wang, and A. G. Fane. Analysis of heat and mass transfer by cfd for performance enhancement in direct contact membrane distillation. *Journal of Membrane Science*, 405-406:38–47, 2012.